

**DESIGN CONCEPTS
OF
RADIO INTERFEROMETERS
USING ARRAYS**

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– AUTHOR –

Introduction

In the last lecture, we have come across the designing concepts of single dish radio telescopes and phased arrays (technically an additive array, where signals from different elements are added after time delay corrections). Yet there is another kind of array known as interferometer array (technically a multiplicative array, where signals from different antenna elements are multiplied after delay corrections). Majority of the radio arrays used for continuum observations are interferometer arrays. The simplest interferometer consists of two antenna antennas. We shall begin here with a simple optical interferometer and then extend its concepts to a simple radio interferometer....

The radio wavelengths are much larger than optical wavelengths. Thus single dish radio telescopes have poor angular resolution as compared to optical telescopes. The resolution can be increased using antenna arrays with large spacings. An optical technique called interferometry consisting of a pair of telescopes was extended to radio for improving the resolution. This uses antenna arrays. We begin with the basic principles from optics.

Recall Raleigh Criterion ...

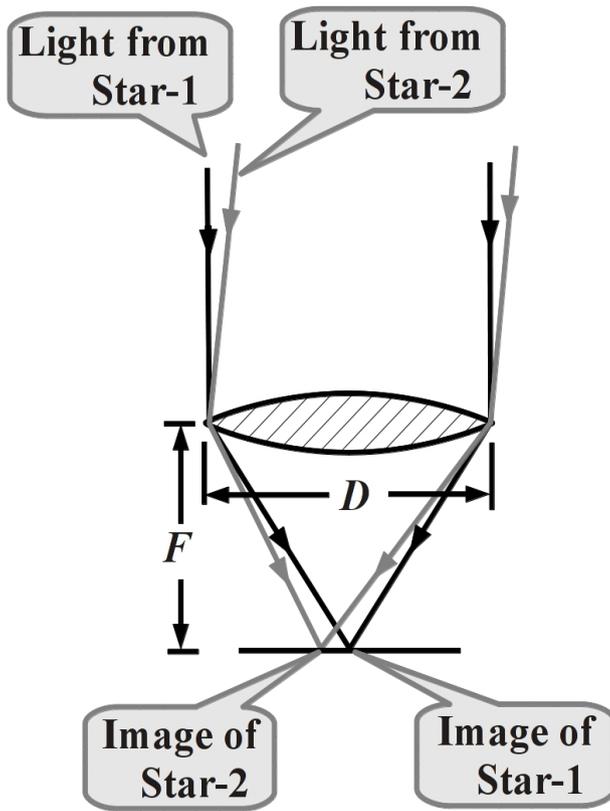
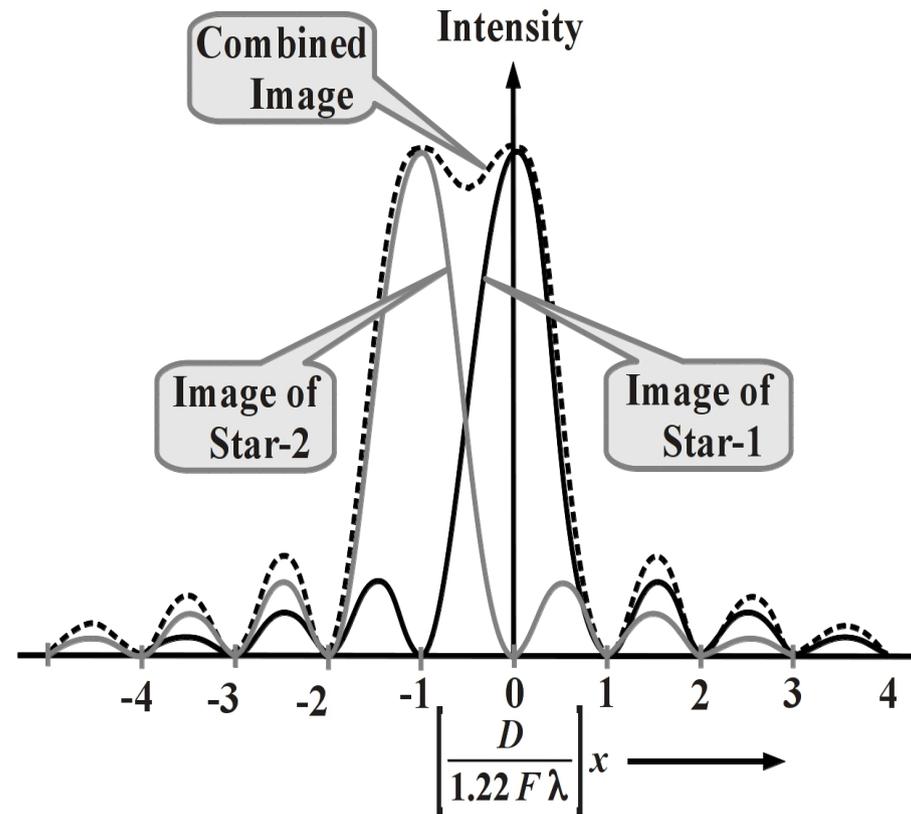


Image from two stars



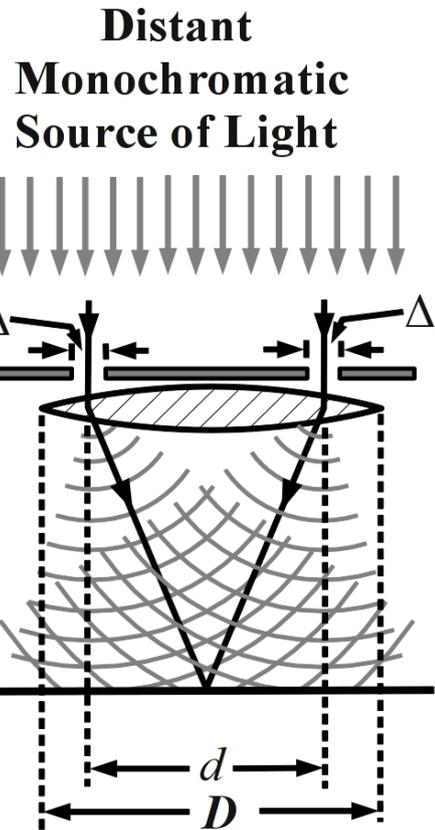
Point spread functions

Rayleigh criterion: Two point sources can be resolved with a single telescope if the angular separation between the two objects is at least equal to or greater than α' as expressed below:

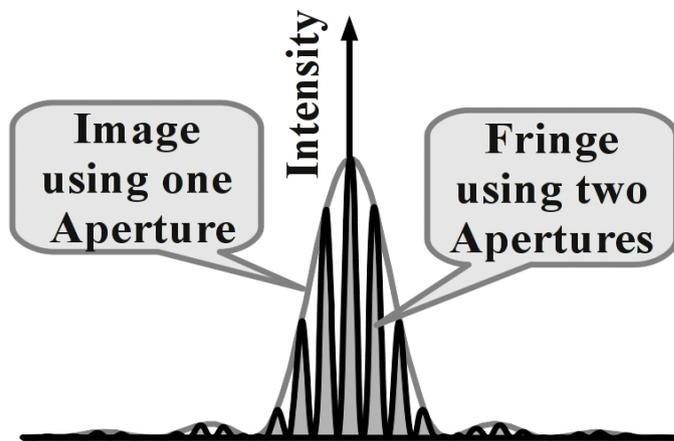
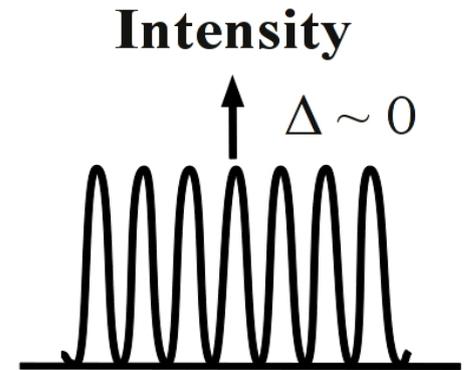
$$\alpha' = 1.22 \frac{\lambda}{D} \approx \frac{\lambda}{D}$$

where, λ is the wavelength and D is the diameter of the lens.

Principles of Optical Interferometer



Consider the geometry shown on the left, where a distant monochromatic point source produces parallel rays near the vicinity of a convex lens of diameter D . The lens is shadowed by a plate having two small holes of diameter Δ . The holes act as apertures. Depending on the distance d between the holes, an interference pattern occurs on the screen.

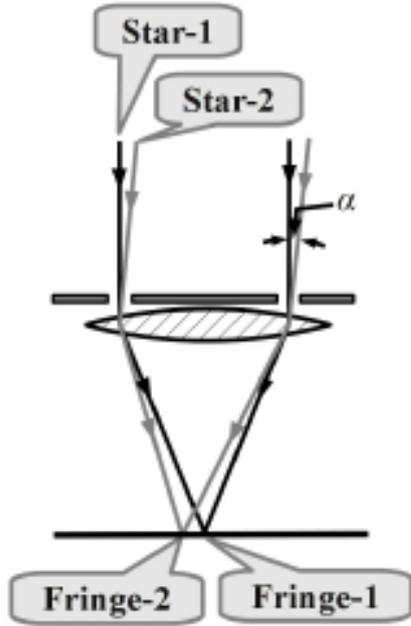


If one of the holes is closed, the fringe vanishes, and the image pattern now takes the shape of the fringe envelope.

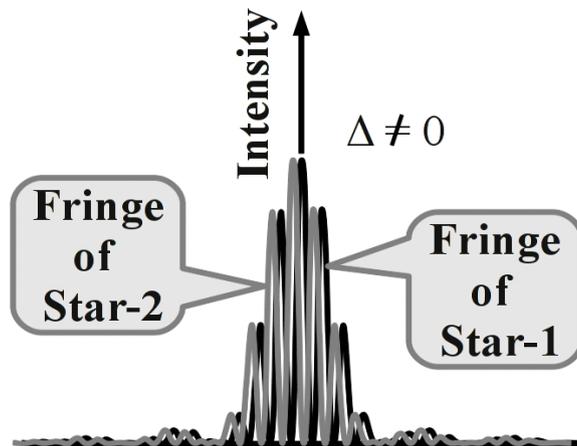
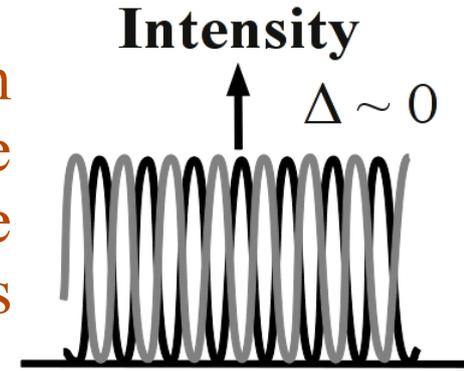
If the apertures are made infinitely small ($\Delta \rightarrow 0$), the fringe becomes uniform. Note that there is a large difference between the maxima and minima resulting in a good contrast.

Principles of Optical Interferometer

Now consider two stars (point sources) of equal intensity and separated by an angle α as shown in left.



If the holes have zero dimension ($\Delta \rightarrow 0$), and if the stars are monochromatic having same wavelength λ , two fringes corresponding to each are formed.



If the holes have non-zero dimension the fringe pattern looks like as illustrated on left.

If both stars are aligned ($\alpha = 0$), the two fringe patterns overlap each other, and resulting intensity is doubled.

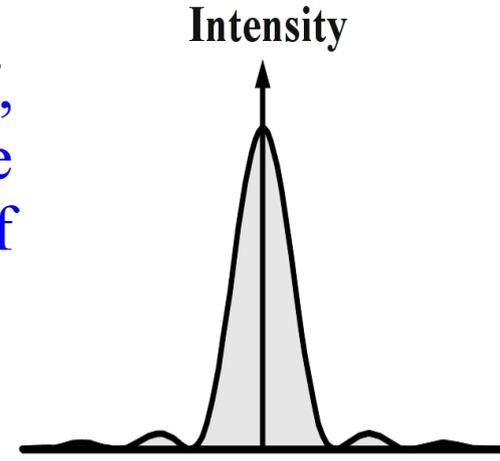
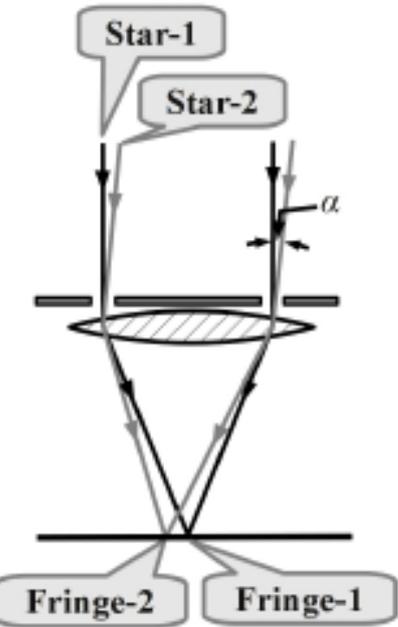
Resolution of Optical Interferometer

The fringes cancel out if $\alpha = \alpha''$, because the maxima of one fringe pattern coincides with the minima of the other, where α'' is given by:

$$\alpha'' = \frac{\lambda}{2d}$$

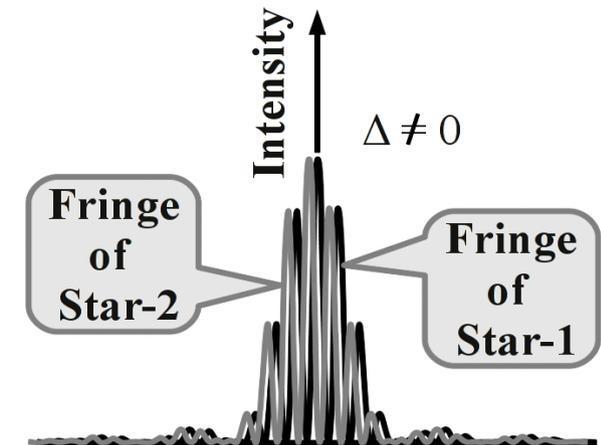
Here, d is the center to center distance between the holes.

The angle α'' is the *resolution of the interferometer* (radians), where λ is the wavelength and d is the center to center distance between the two holes. Note that d and λ should be in same units (length).



Note:

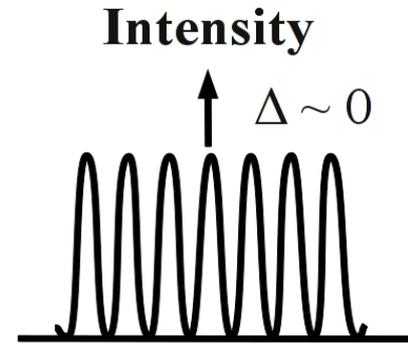
If α is increased beyond α'' , the fringes reappear and disappear In general, the fringes are clearly visible when $\alpha = 2n\alpha''$, where n is an integer. The fringes disappear when $\alpha = (2n+1)\alpha''$.



Applying Raleigh Criterion ...

The angular separation of two monochromatic point sources at which the corresponding fringe patterns are mutually displaced by half a fringe is:

$$\alpha'' = \frac{\lambda}{2d}$$

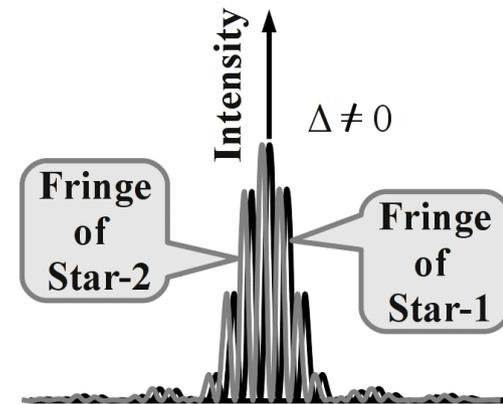


If the apertures are on the lens edges ($d = D$), then

$$\alpha'' = \frac{1}{2} \left(\frac{\lambda}{D} \right) = \frac{1}{2} \left(\frac{\alpha'}{1.22} \right) = 0.409 \alpha' \approx \frac{\alpha'}{2}$$

where α' is the resolution of the lens aperture (Raleigh criterion):

$$\alpha' = 1.22 \frac{\lambda}{D}$$



Hence resolution is doubled, but aperture size is reduced. So two separate lens are used with an arrangement of mirrors to combine their outputs. Increasing the distance d of separation between the two apertures increases the angular resolution at any particular wavelength.

In a radio interferometer, the two antennas may be considered as the two lenses. Hence the resolution of a radio interferometer is:

$$\alpha'' = \frac{\lambda}{2d}$$

Visibility (Fringe Contrast)

The fringe pattern can be seen as long as contrast between bright and dark areas of the fringe are good. It depends on the intensity differences between fringe maxima and fringe minima. A dimensionless quantity known as *visibility* \mathcal{V} (also known as *fringe contrast*) whose magnitude is expressed as:

$$|\mathcal{V}| = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

where I_{\max} and I_{\min} respectively are two neighboring maxima and minima within the fringe. It quantifies the fringe contrast. The image possesses maximum contrast if $I_{\min} = 0$. Under these conditions, $|\mathcal{V}| = 1$. No fringe is visible when $I_{\min} = I_{\max}$, which is indicated by $\mathcal{V} = 0$.

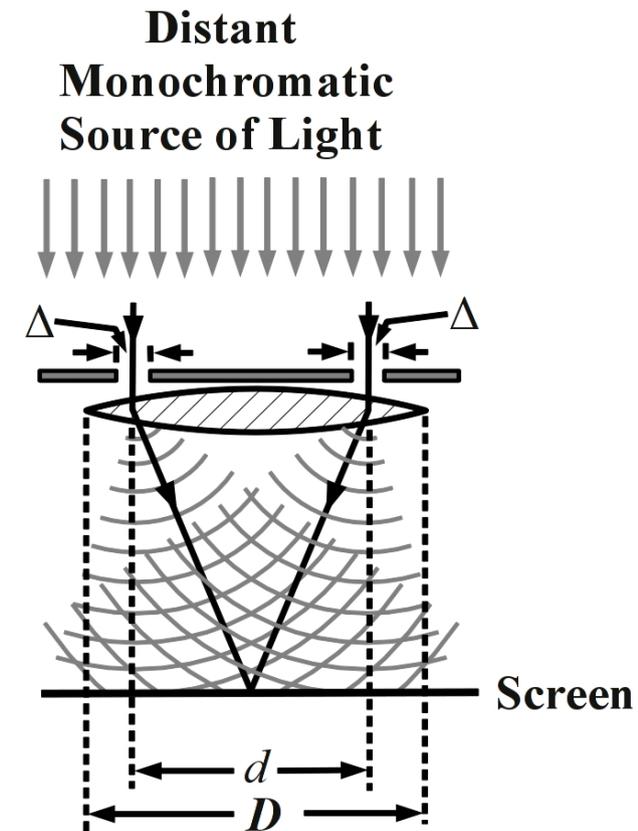
Note: Visibility \mathcal{V} is a complex number and its magnitude always lies between 0 and 1. The fringe is visible if and only if the magnitude of visibility is greater than 0. The fringe becomes more and more distinguishable as the magnitude of \mathcal{V} increases.

Coherence Length

Irrespective of source distance or wavelength λ , the brightest spot of the fringe always appear at the center of the image. However, the distance between a maxima and its neighboring minima depends on λ . If the source is polychromatic, a number of frequency components reach the screen whose phases differ depending on their wavelengths. Each of these interferes which effectively reduces the fringe contrast \mathcal{V} . The path difference between two waves which results $\mathcal{V} = 0$ is known as coherence length l_c . It is given by:

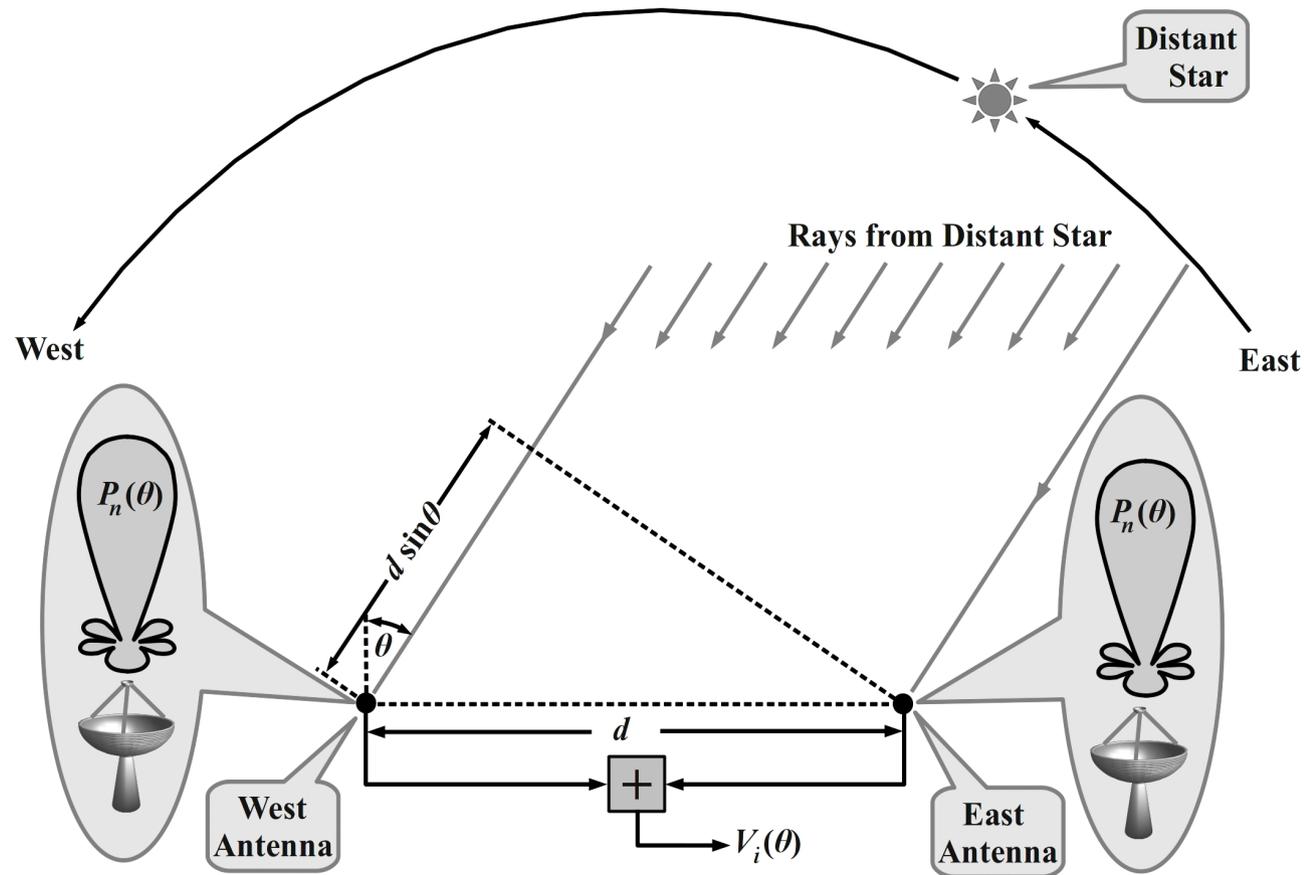
$$l_c = \frac{c}{n_r \Delta\nu} \approx \left(\frac{1.4}{\pi n_r} \right) \frac{\lambda^2}{\Delta\lambda}$$

where, c is the speed of light, n_r is the refractive index of the medium, $\Delta\nu$ is the band-width of light, λ is the wavelength, and $\Delta\lambda$ is the wave-band of the source (also known as *spectral width*).



Additive Radio Interferometer

Two antennas having identical characteristics are separated by a distance d and located near the equator along the East-West line. The main beams are towards zenith. As a radio star moves from East to West, the instantaneous outputs when added forms a fringe pattern as a function of time or angle θ (see next page).

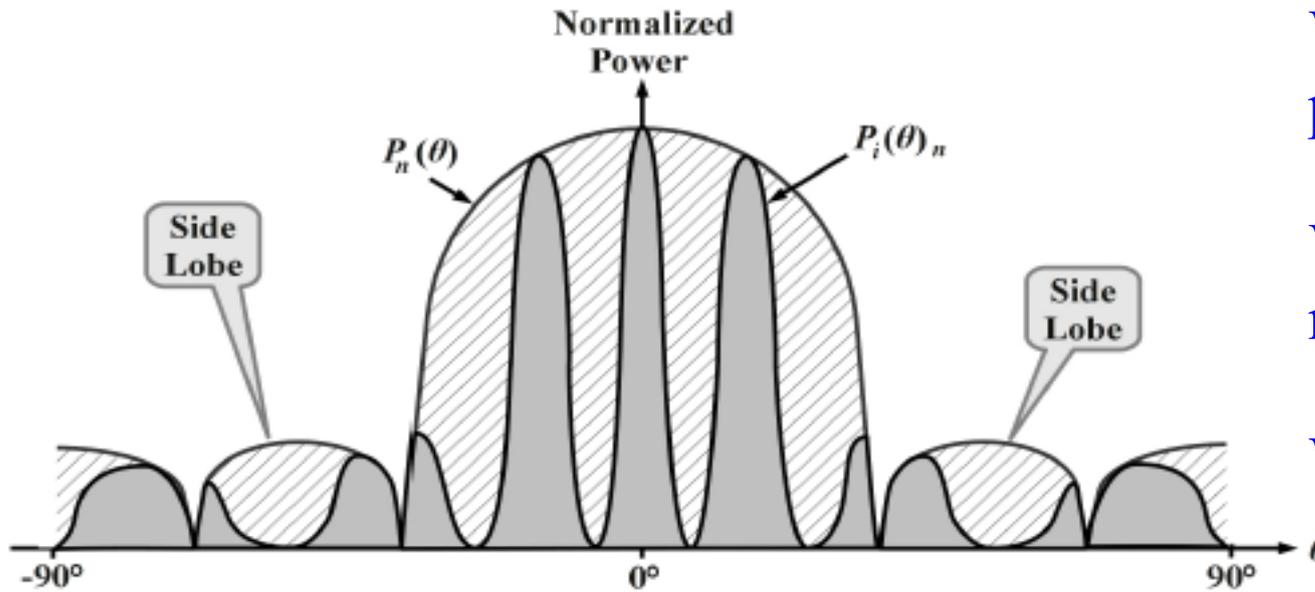


Separation length d between the two is known as **baseline**. In terms of wavelength λ , the baseline is $d_\lambda = d/\lambda$. It resembles to a two element grating array. If the star is on the eastern sky, the wave-fronts first reach the East antenna, and after a time delay τ they reach the West antenna.

$$\tau = \frac{d \sin \theta}{c}$$

The adder output voltage V_i is a function of θ .

Fringe pattern of Additive Interferometer



When the source rises, τ is positive maximum:

$$\tau = d/c$$

When the source sets, τ is negative maximum:

$$\tau = -d/c$$

When the source is at zenith,

$$\tau = 0$$

Fringe pattern (normalized output power vs. θ).

$P_n(\theta)$ - normalized power pattern of the antennas.

$P_i(\theta)_n$ - normalized power output from the adder (fringe pattern).

Antenna beam patterns scale the incoming signals. Adder power output has a modulation of antenna beam pattern (along θ). $P_i(\theta)_n$ is not equally spaced along θ . Relative delay increases with magnitude of θ , and acquires maxima at $\theta = \pm 90^\circ$. When the star is at zenith ($\theta = 0$), $\tau = 0$ and the adder power peaks.

Improvements: Antennas may track the source to avoid fringe modulation (by beam pattern). Before adding the signals, their phases should be corrected by introducing artificial time delay (*instrumental time delay*) so as to maximize the adder output. LNAs (next to antennas) can improve SNR.

Multiplicative Radio Interferometer

- Antennas track the source.
- Instrumental delay τ_i is on.
- Signals are multiplied.

Geometric delay τ_g
 ($\tau_g = d \sin\theta / c$) is
 balanced by
 instrumental delay τ_i .

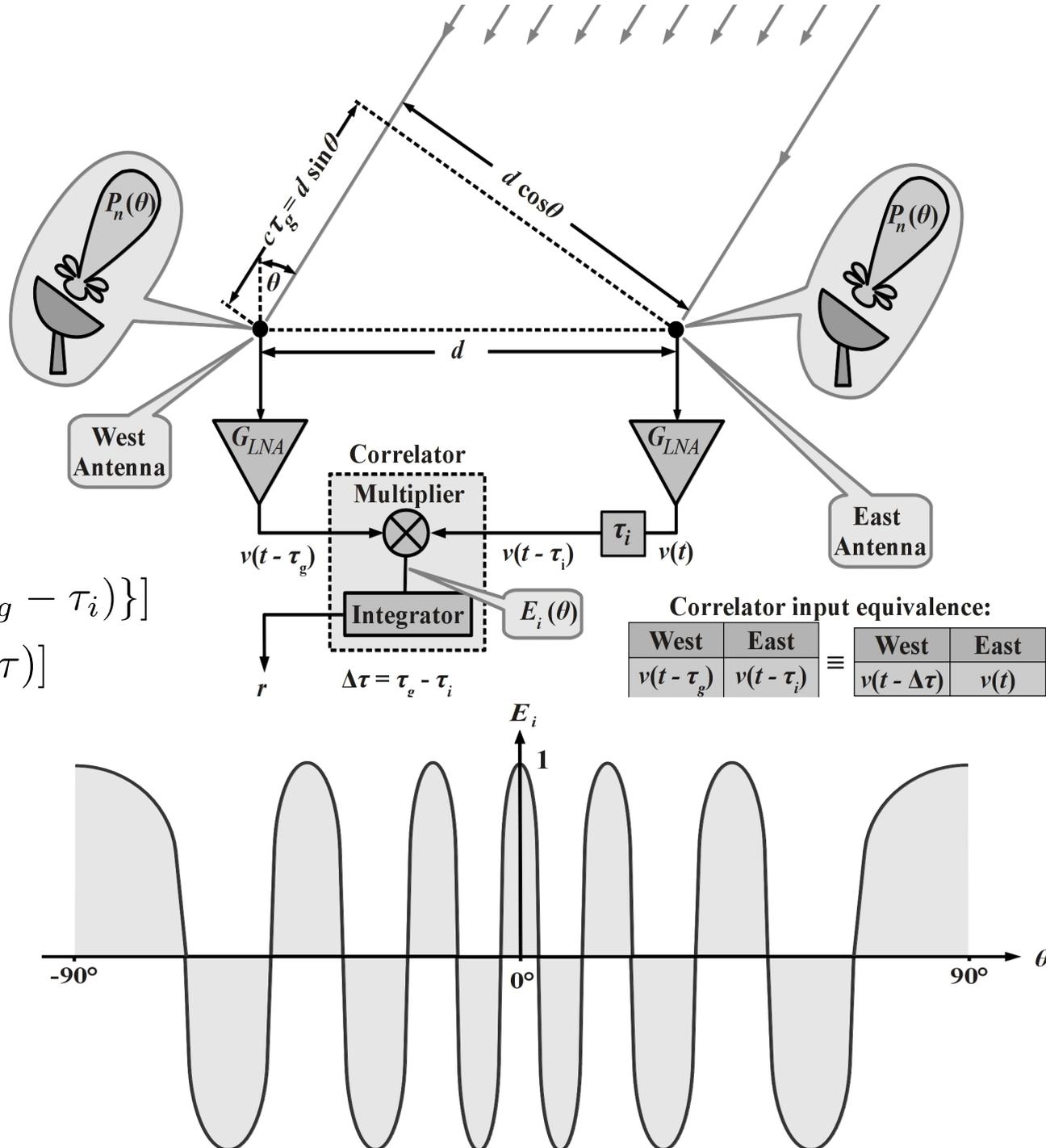
Multiplier Output:

$$E_i = \sin(2\pi\nu t) \sin[2\pi\nu \{t - (\tau_g - \tau_i)\}]$$

$$= \sin(2\pi\nu t) \sin[2\pi\nu (t - \Delta\tau)]$$

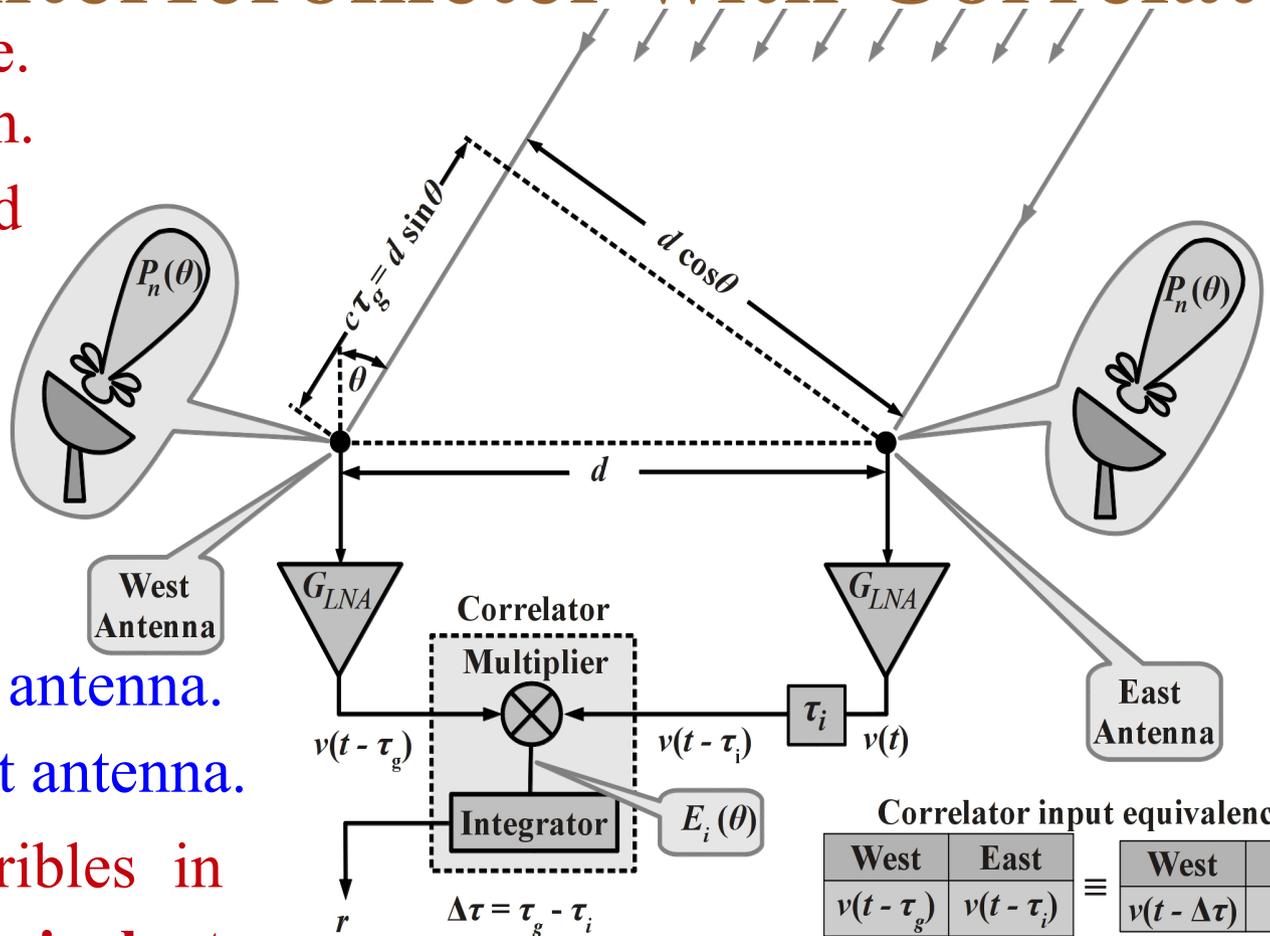
where, $\Delta\tau = \tau_g - \tau_i$

Right: Tentative sketch of multiplier output $E_i(\theta)$ for the case when no instrumental delay correction is applied, i.e., $\tau_i = 0$.



Multiplicative Interferometer with Correlator

- Antennas track the source.
- Instrumental delay τ_i is on.
- Signals are multiplied and integrated (correlator backend).
- Frontend amplifiers with gain G_{LNA} are added.



Correlator inputs:

$v(t - \tau_i)$ - Voltage from East antenna.

$v(t - \tau_g)$ - Voltage from West antenna.

Adding τ_i to the time variables in above we obtain the **Equivalent correlator inputs as:**

$v(t)$ - Voltage from East antenna.

$v(t - \Delta\tau)$ - Voltage from West antenna, where $\Delta\tau = \tau_g - \tau_i$

Correlator output:

(integration over $2T$ secs)

$$r(\Delta\tau) = \frac{1}{2T} \int_{-T}^T v(t) v(t - \Delta\tau) dt$$

$2T$ varies from several milliseconds to few seconds. It should be chosen much larger than reciprocal of the band-width $\Delta\nu$, where ν is frequency.

Interferometer with Correlator continued ...

The upper limit of integration time is the duration of observation, which can be hours. Thus, for a very large integration, we may express $r(\Delta\tau)$ as:

$$r(\Delta\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T v(t) v(t - \Delta\tau) dt$$

Recall the Fourier transform Wiener-Khinchin relationship between square of voltage spectrum (power spectrum) $S(\nu)$ and the auto-correlation $R(\tau)$:

$$S(\nu) = \int_{-\infty}^{\infty} R(\tau) e^{-j2\pi\nu\tau} d\tau$$

Let $H(\nu)$ also be a voltage spectrum. Then $|H(\nu)|^2$ represents the power spectrum. The correlator output $r(\Delta\tau)$ can be related to $|H(\nu)|^2$ as:

$$|H(\nu)|^2 = \int_{-\alpha}^{\alpha} r(\Delta\tau) e^{-j2\pi\nu\Delta\tau} d\Delta\tau$$

Its inverse relationship is $r(\Delta\tau) = \int_{-\alpha}^{\alpha} |H(\nu)|^2 e^{j2\pi\nu\Delta\tau} d\nu$

For a wide-band radio noise source, $H(\nu)$ resembles the pass-band shape of the receiving system. Thus, ***interferometer output as a function of time delay difference $\Delta\tau$ is a Fourier transform of cosmic signal's power spectrum.*** The band-width of correlated cosmic signal is same as of the amplifiers.

Fringe Stop in Correlator Interferometer

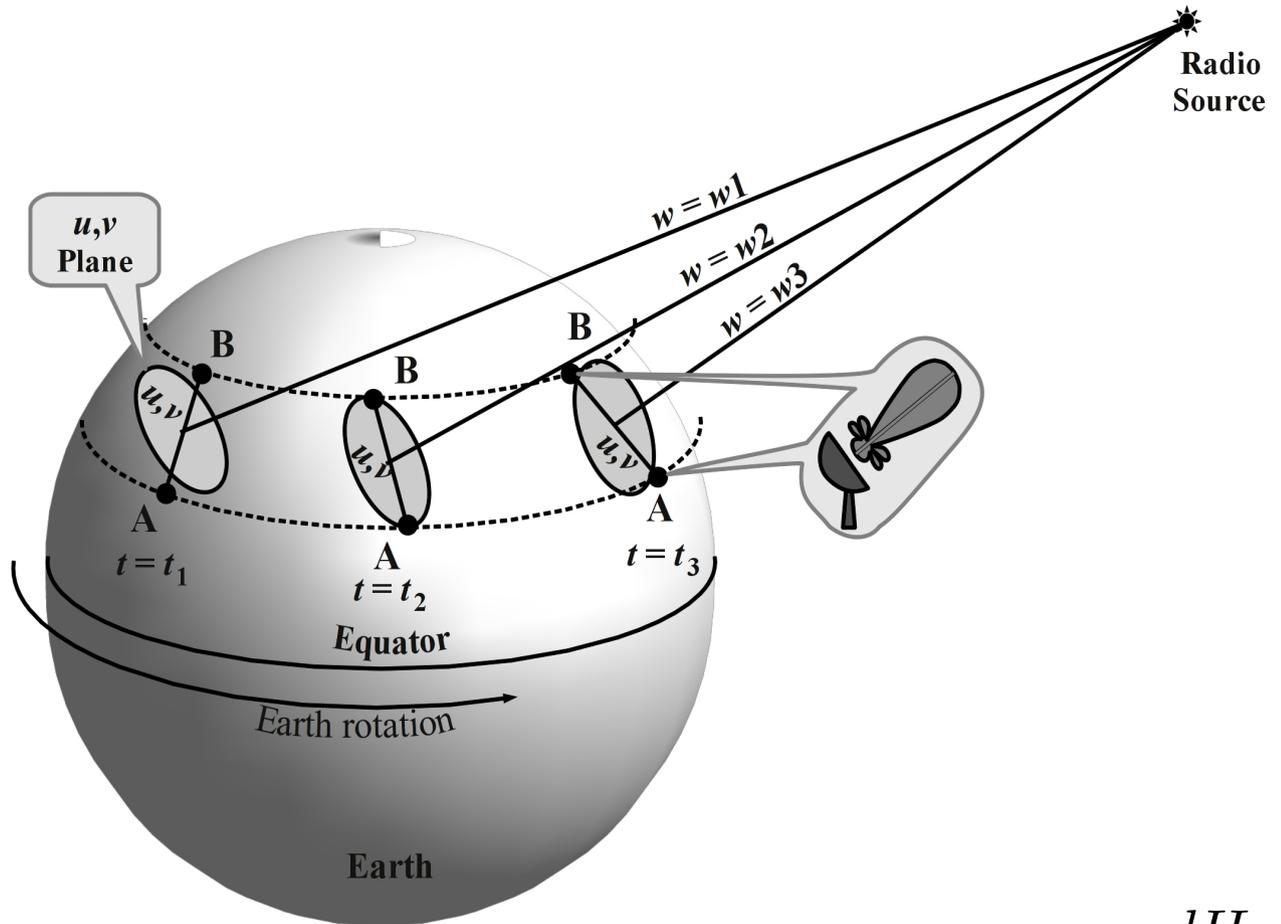
Let the antennas be tracking a source. Propagation delay $d \sin\theta/c$ keeps changing with Earth rotation. If we compensate the geometrical delay $\tau_g = d \sin\theta/c$ by equal amount of instrumental delay τ_i such that $\Delta\tau = \tau_g - \tau_i = 0$, then signals entering the correlator are in phase and interferometer output peaks. This point of observation on the sky is called the fringe stopping center.

Propagation or geometric delay $\tau_g = d \sin\theta/c$ keeps changing with time. It is possible to evaluate the time rate of change of τ_g . If we do not compensate τ_g with τ_i , the fringe output will vary with time. This frequency is called the fringe frequency ν_{fringe} . Please note that ν_{fringe} gets scaled with the type of correlator receiving system used. For example, the correlator system may be a heterodyne type with variants like lower side band, upper side band, double side band etc. For a correlator system without any IF conversion, the fringe frequency ν_{fringe} is expressed as time derivative of w :

$$\nu_{fringe} = \frac{d w}{dt} \quad \text{where, } w \text{ belongs to the } u, v, w \text{ coordinate system.}$$

Fringe Stop in Correlator Interferometer

Two antennas A and B located at two different longitude-latitudes pair form an interferometer. Relative to a radio source, the correlator baseline AB are shown for three different time instants t_1, t_2, t_3 , where $t_1 < t_2 < t_3$. Projection of AB on $u-v$ plane and source distance along w change with Earth rotation.



Angular frequency of Earth rotation ω_e is related to hour angle H as: $\omega_e = \frac{dH}{dt}$

With δ as source declination, we may expand fringe frequency as:

$$\nu_{fringe} = \frac{dw}{dt} = \frac{dw}{dH} \frac{dH}{dt} = \frac{dw}{dH} \omega_e = -\omega_e u \cos \delta$$

In a RF correlator, $w = c\tau_g$. Hence, $\frac{d\tau_g}{dt} = \frac{1}{c} \frac{dw}{dt} = \frac{\nu_{fringe}}{c} = -\frac{\omega_e u \cos \delta}{c}$

Fringe Stop in Correlator Interferometer

We have seen:

(i) Fringe frequency depends on angular frequency of Earth rotation.

$$\nu_{fringe} = \frac{dw}{dt} = \frac{dw}{dH} \frac{dH}{dt} = \frac{dw}{dH} \omega_e = -\omega_e u \cos \delta$$

(ii) Geometric delay continuously changes with Earth rotation.

$$\frac{d\tau_g}{dt} = -\frac{\omega_e u \cos \delta}{c}$$

Interferometers should therefore cancel the geometric delay τ_g with equal amount of instrumental delay τ_i to remove the effect of Earth rotation. In other words, interferometers should achieve the *fringe stop* condition.

Note: Modern interferometers use heterodyne systems. They compensate τ_g at the IF (not at RF). Hence, visibility at any direction other than the fringe stop center changes slowly due to the rotation of the baseline with respect to the source. Integrator time constant should be short enough to observe these variations. *If delay is compensated in the RF stage, fringe stopping is possible in any direction.*

Gaussian Passband response: Corr. Inter.

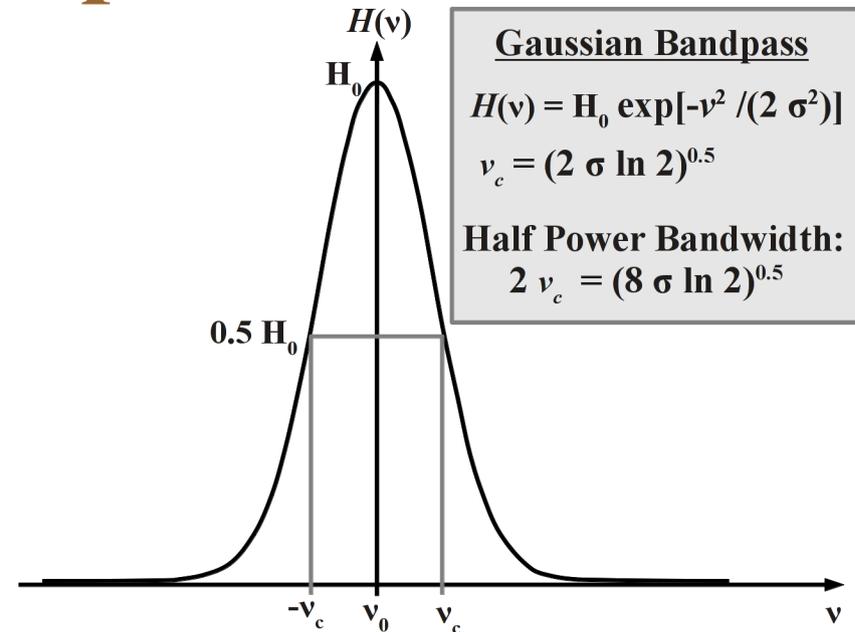
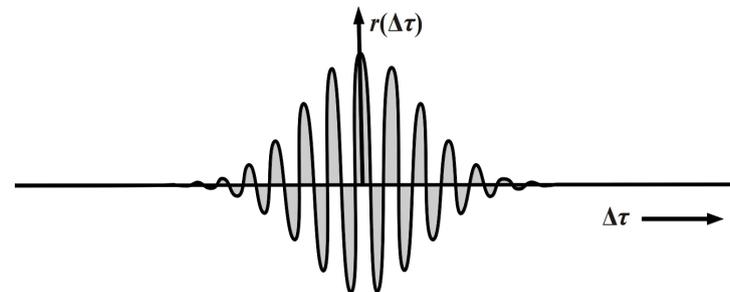
Assume the amplifier pass-bands as Gaussian shaped, centered at a frequency ν_0 and have a band-width factor σ . Hence their band-width at half maxima is $(8 \ln 2 \sigma)^{0.5}$ as shown. The power spectrum can be expressed as a sum of two Gaussian functions (for left and right halves of the filter response around ν_0):

$$|H(\nu)| = \frac{1}{2\sigma\sqrt{2\pi}} \exp\left[-\frac{(\nu - \nu_0)^2}{2\sigma^2}\right] + \frac{1}{2\sigma\sqrt{2\pi}} \exp\left[-\frac{(\nu + \nu_0)^2}{2\sigma^2}\right]$$

We also know that:
$$r(\Delta\tau) = \int_{-\infty}^{\infty} |H(\nu)|^2 \exp(j2\pi\nu\Delta\tau) d\nu$$

Substituting $|H(\nu)|$ we get:
$$r(\Delta\tau) = \exp(-2\pi^2\tau^2\sigma^2) \cos(2\pi\nu_0\Delta\tau)$$

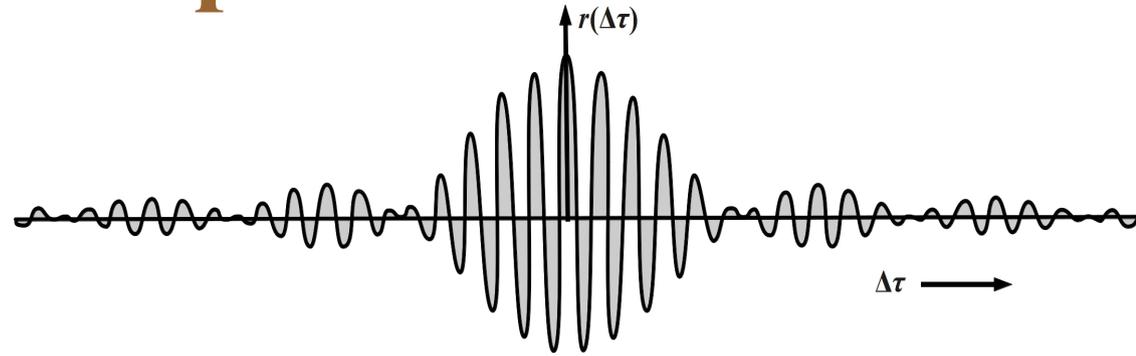
The response is shown. Note that $r(\Delta\tau)$ is a cosine wave modulated by a Gaussian envelope.



Gaussian Passband Response continued ...

Note:

For a rectangular passband, the correlator output is shown on right:



The Gaussian response was:

$$r(\Delta\tau) = \exp(-2\pi^2\tau^2\sigma^2) \cos(2\pi\nu_0\Delta\tau)$$

Now consider the case when geometric delay is not compensated by instrumental delay, i.e., $\tau_i = 0$ so that $\Delta\tau = \tau_g = d \sin \theta/c$.

Substituting $\Delta\tau = \tau_g = d \sin \theta/c$ in $r(\Delta\tau)$ we get:

$$r(\tau_g) = \exp\left\{-2\left(\frac{\pi d\sigma}{c} \sin \theta\right)^2\right\} \cos\left(\frac{2\pi\nu_0 d}{c} \sin \theta\right)$$

We find that wide band-widths (large σ) and large baselines (large d) result in narrow fringe envelopes if delay is not compensated.

The fringe envelope is considered nuisance for most mapping applications, except in VLBI. Generally, one uses the fringe where its amplitude is optimum. It is done by always maintaining τ_i equal to τ_g such that $\Delta\tau \rightarrow 0$. For practical purposes, τ_i is adjusted in time steps of the order of the ν_0^{-1} .

Simple Correlators

Input (volts):

$$v_1(t), v_2(t)$$

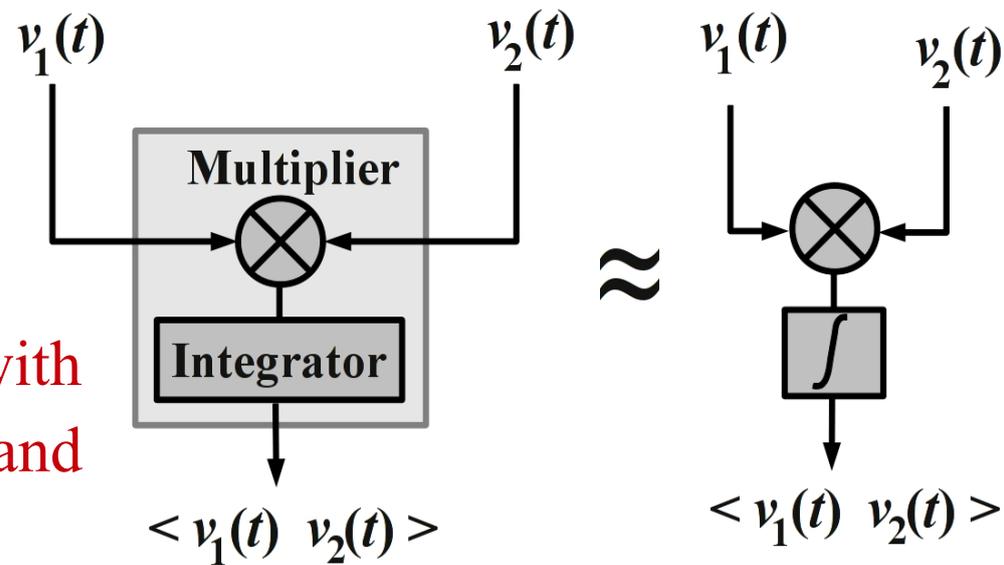
Output (volts square, time averaged):

$$r(\tau) = \langle v_1(t) v_2(t) \rangle$$

Let $v_1(t)$ and $v_2(t)$ be cosine waves with frequency ν and having phase shifts α_1 and α_2 . and amplitudes A_1 and A_2 respectively.

$$v_1(t) = A_1 \cos(2\pi\nu t + \alpha_1)$$

$$v_2(t) = A_2 \cos(2\pi\nu t + \alpha_2)$$



A simple correlator.

The product of $V_1(t)$ and $V_2(t)$ before integration is:

$$P_{12}(t) = \frac{1}{2} A_1 A_2 \{ \cos(4\pi\nu t + \alpha_1 + \alpha_2) + \cos(\alpha_1 - \alpha_2) \}$$

Since integrator acts as a low pass filter, it removes the higher frequency components. Thus the output after integration is:

$$r(\tau) = \langle v_1(t) v_2(t) \rangle = \frac{1}{2} A_1 A_2 \cos(\alpha_1 - \alpha_2)$$

Thus we see that correlator output $r(\tau)$ depends on the phase difference between the two signals. *The output is maximum when the phase difference is zero.*

Complex Correlators

Inputs:

$v_1(t)$ and $v_2(t)$ - Cosine waves with frequency ν , having phase shifts α_1 and α_2 and amplitudes A_1 and A_2 respectively.

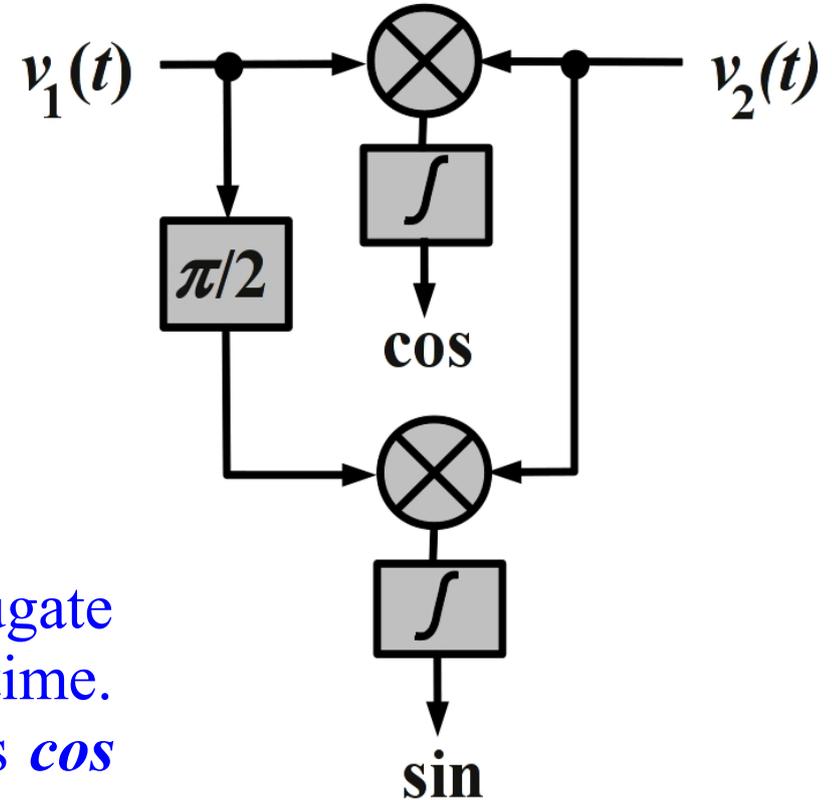
$$v_1(t) = A_1 e^{j(2\pi\nu t + \alpha_1)}$$

$$v_2(t) = A_2 e^{j(2\pi\nu t + \alpha_2)}$$

Outputs:

Product of one signal with complex conjugate of the other and integrated over a period of time. It contains real and imaginary parts known as *cos* and *sin*.

$$\begin{aligned} v_1(t) v_2^*(t) &= A_1 A_2 e^{j(2\pi\nu t + \alpha_1)} e^{-j(2\pi\nu t + \alpha_2)} \\ &= A_1 A_2 [\cos(\alpha_1 - \alpha_2) + j \sin(\alpha_1 - \alpha_2)] \end{aligned}$$



A complex correlator.

The real output (**cos**) depends only on the phase difference. The imaginary output (**sin**) also depends on phase difference but has an added phase difference of $\pi/2$ in one of the two signals. Complex correlators give information about both amplitude and phase. Since both channels are independent of each other, noise immunity of a complex correlator is $\sqrt{2}$ times better.

Correlator Arrays, Cross-Correlator

A correlator array with four antennas (single polarization) are shown on right.

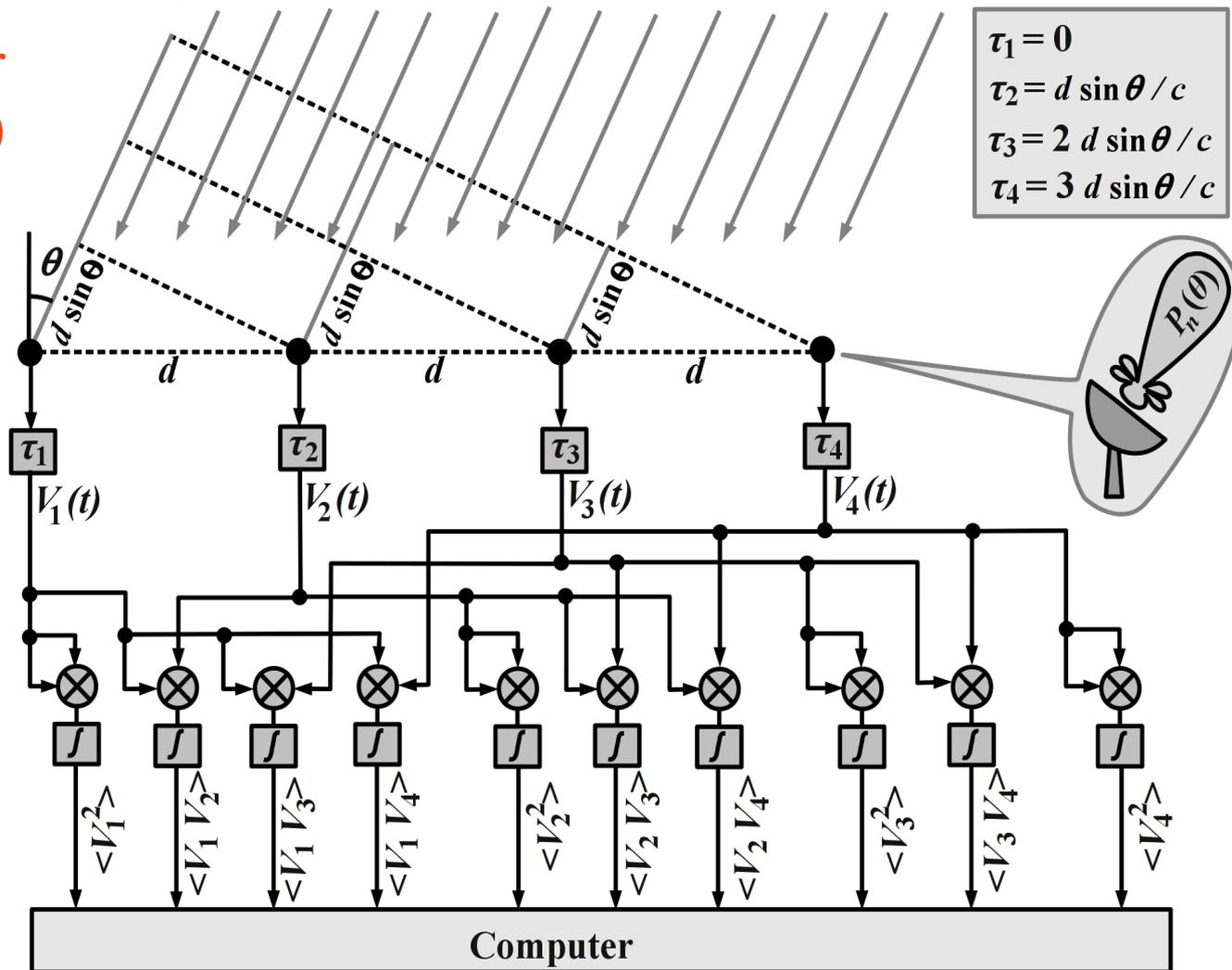
No. of self-correlations:

It is same as number of antennas n_a .

No. of cross-correlations n_c :

It is ${}^{na}C_2$ expressed as:

$$n_c = {}^{na}C_2 = n_a \frac{(n_a - 1)}{2}$$



If the self-correlations are removed, it is called a **cross-correlator**, and the instantaneous sensitivity of the array reduces by a factor κ_c .

$$\kappa_c = 1 - \frac{1}{n_a}$$

This loss becomes negligible if $n_a \gg 1$. Thus for a large array, radio maps can be constructed from cross terms alone without degrading its quality.

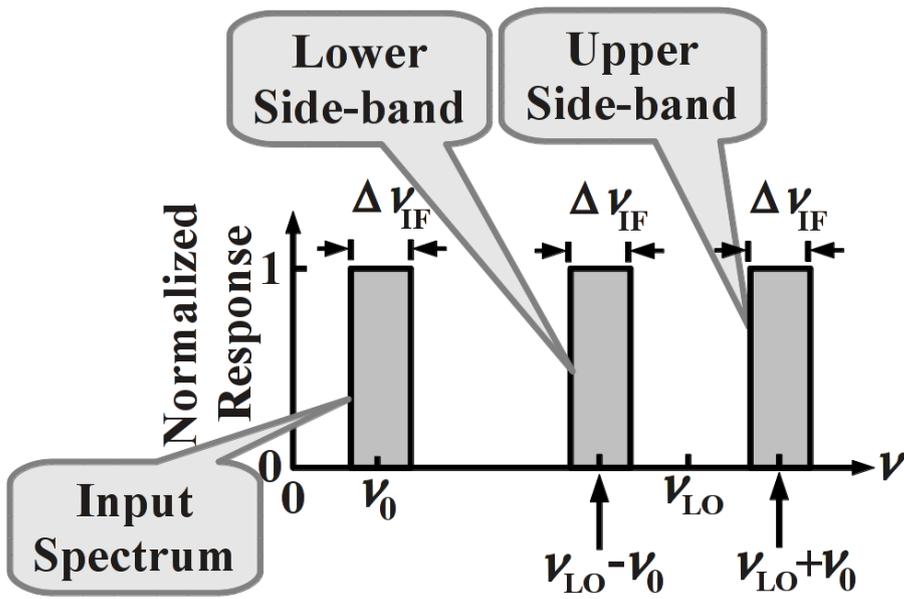
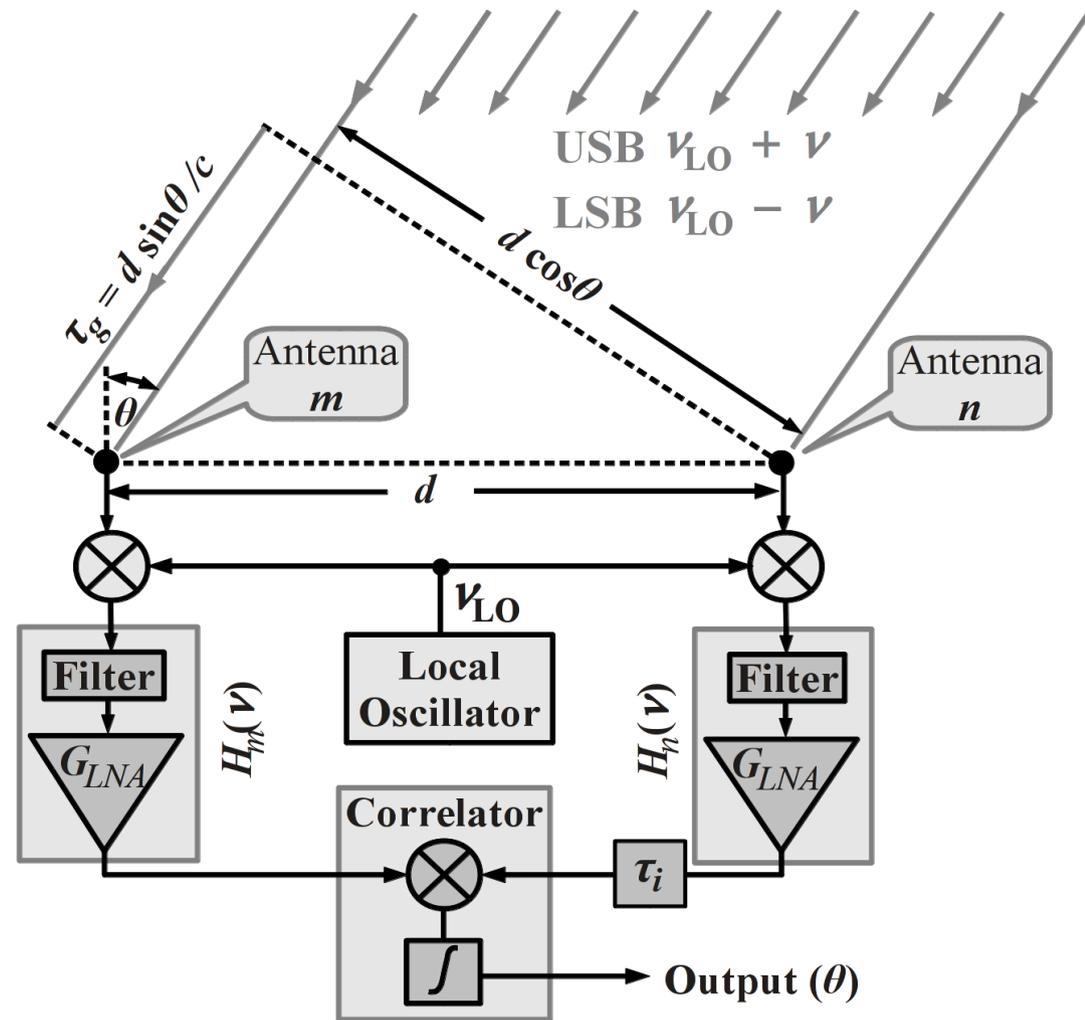
Cross-Correlator Arrays vs. Phased Arrays

As said before, if we remove the self components from the correlator array, it becomes a *cross-correlator*. We now compare it with a phased array:

- The response of a point source using a cross-correlator array is same as of a phased array, except for missing self terms in the former.
- If the source is extended, cross-correlator gives complete information, whereas phased array catch signals at equal phase intervals within the main beam and side-lobes.
- Since the self terms are absent, the cross-correlator output can be positive and negative, whereas the phased array output is always positive for it uses a square law detector.
- The signal to noise ratio obtained from cross-correlation is better than of a phased array. The reason being, correlator gathers the coherent signals. The noise being incoherent from different antenna, they tend to diminish. Hence cross terms are more important than self terms. A phased array adds both noise and signals, so the noise cancellation is less.

Cross-Correlator: Receiving Systems

Right: A heterodyne correlator receiving system using two antennas of an array. The geometric delay τ_g is nullified by instrumental delay τ_i continuously while tracking the source. $H_m(\nu)$ and $H_n(\nu)$ are the overall band-pass characteristics of the amplifiers and filters in the signal path.



Left: Band-pass characteristics of the input signal and mixer outputs with reference to LO frequency ν_{LO} and input center frequency ν_0 .

Cross-Correlator: Single Side Band Response

Let ϕ_m be phase change in the signal path from antenna m to correlator input resulting from τ_g and LO. Let ϕ_n be the corresponding phase change in the signal path from antenna n to correlator (including the instrumental delay τ_i). With reference to the (u, v, w) coordinate system, we express the visibility $\mathcal{V}(u, v)$ in its magnitude and phase form in the u - v plane as:

$$\mathcal{V}(u, v) = |\mathcal{V}| e^{j\phi_v} \quad \text{where } \phi_v \text{ is the phase of the visibility.}$$

Within a pass-band $d\nu$, the infinitesimal correlator response dr will be:

$$dr = \text{Re} \left[A_e |\mathcal{V}| H_m(\nu) H_n^*(\nu) e^{j(\phi_n - \phi_m - \phi_v)} d\nu \right]$$

Here, A_e is the effective aperture area of any one antenna (same for both) looking at the source, and $H_m(\nu)$ and $H_n(\nu)$ are the response of the filter amplifier combinations. Note that $H_n^*(\nu)$ is the complex conjugate of $H_n(\nu)$.

The correlator response over the entire system pass-band is the integration of dr over the entire band-width:

$$r = \text{Re} \left[A_e |\mathcal{V}| \int_{-\infty}^{\infty} H_m(\nu) H_n^*(\nu) e^{j(\phi_n - \phi_m - \phi_v)} d\nu \right]$$

Cross-Corr.: USB with single IF conversion

Let the filters allow only USB frequencies ($\nu_{\text{LO}} + \nu$) to pass. Due to geometric delay τ_g , signals from antenna m undergoes a phase shift of $2\pi(\nu_{\text{LO}} + \nu) \tau_g$ when it reaches the mixer. Let θ_m be the negative phase contribution from LO to the signal from antenna m in the mixer. Similarly, let θ_n be the negative phase contribution from LO to the signal from antenna n . The overall phase changes ϕ_m and ϕ_n respectively for the paths from m and n antennas to the mixer are:

$$\phi_m = -2\pi(\nu_{\text{LO}} + \nu) \tau_g - \theta_m \quad \text{and} \quad \phi_n = -2\pi\nu\tau_i - \theta_n$$

Substituting the above in $r = \text{Re} \left[A_e |\mathcal{V}| \int_{-\infty}^{\infty} H_m(\nu) H_n^*(\nu) e^{j(\phi_n - \phi_m - \phi_\nu)} d\nu \right]$

we obtain the USB correlator output r_{usb} as:

$$r_{\text{usb}} = \text{Re} \left[A_e |\mathcal{V}| e^{j[2\pi\nu_{\text{LO}}\tau_g + (\theta_m - \theta_n - \phi_\nu)]} \int_{-\infty}^{\infty} H_m(\nu) H_n^*(\nu) e^{j2\pi\nu\Delta\tau} d\nu \right]$$

where, $\Delta\tau = \tau_g - \tau_i$ is difference between geometric and instrumental delays.

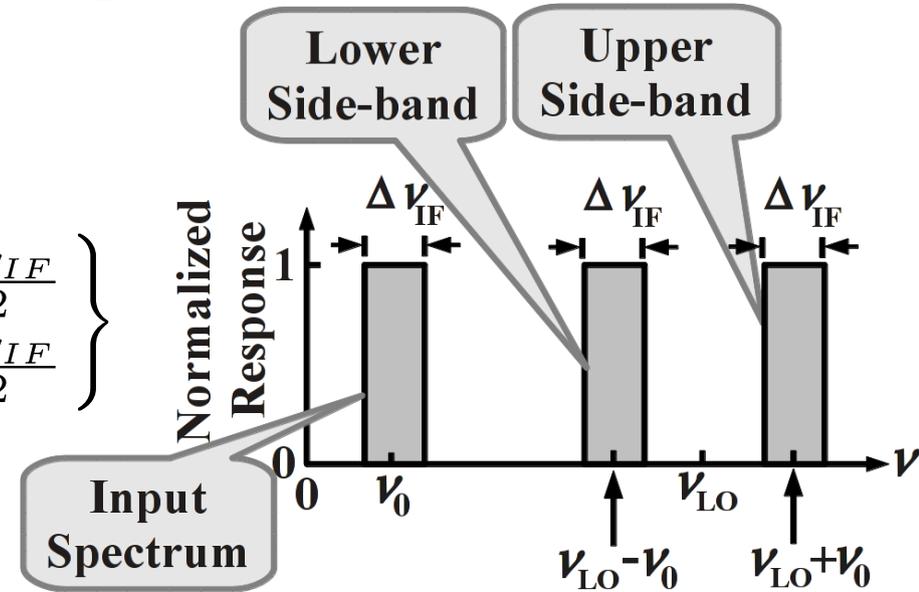
If source is sufficiently close to center of observing field, the above integral becomes half of the Fourier transform of the cross power spectrum $H_m(\nu) H_n^*(\nu)$. Since $\Delta\tau$ is small but non-zero, it introduces a linear phase change across the band.

Cross-Corr.: USB with single IF conversion

Let the magnitudes of $H_m(\nu)$ and $H_n(\nu)$ be identical and flat across the band-width as:

$$|H_m(\nu)| = |H_n(\nu)| = \begin{cases} H_0, & |\nu - \nu_0| < \frac{\Delta\nu_{IF}}{2} \\ 0, & |\nu - \nu_0| > \frac{\Delta\nu_{IF}}{2} \end{cases}$$

where, $\Delta\nu_{IF}$ is the IF filter band-width.



Defining $G_{mn} = |G_{mn}| e^{j\phi_G}$ as instrumental gain factor, where ϕ_G is the phase difference between the two signal paths produced by the filter-amplifier combinations excluding the LO phase contributions (θ_m and θ_n). These will be added separately whenever required.

For non-zero $\Delta\tau$,

$$|G_{mn}(\Delta\tau)| e^{j(2\pi\nu_0\Delta\tau + \phi_G)} = A_e \int_{-\infty}^{\infty} H_m(\nu) H_n^*(\nu) e^{j2\pi\nu\Delta\tau} d\nu$$

Substituting the above equation in the equation of r_{usb} (previous slide) we get:

$$r_{usb} = |\mathcal{V}| |G_{mn}(\Delta\tau)| \cos [2\pi (\nu_{LO} \tau_g + \nu_0 \Delta\tau) + (\theta_m - \theta_n) - \phi_v + \phi_G]$$

Cross-Corr.: USB with single IF conversion

From last slide, we have the USB correlator output (r_{usb}) as:

$$r_{\text{usb}} = |\mathcal{V}| |G_{mn}(\Delta\tau)| \cos [2\pi (\nu_{\text{LO}} \tau_g + \nu_0 \Delta\tau) + (\theta_m - \theta_n) - \phi_v + \phi_G]$$

The $2\pi\nu_0\tau_g$ term inside the cosine function produces a quasi-sinusoidal variation of r_{usb} with the motion of source. The overall phase depends on:

- (i) Phase responses of the signal channels.
- (ii) Delay error $\Delta\tau$.
- (iii) Phase of the visibility function.
- (iv) Relative phases of the LO signals.

Since the geometrical delay τ_g appears at the RF, whereas the compensating instrumental delay τ_i is applied in IF, the output oscillates. The oscillation frequency (natural fringe frequency) ν_{fringe} at the output is:

$$\nu_{\text{fringe}} = \nu_{\text{LO}} \frac{d\tau_g}{dt}$$

Note: Since the frequencies at RF and IF differ by ν_{LO} , the contribution to ν_{fringe} comes from ν_{LO} as the source changes its position on the sky.

Cross-Corr.: LSB with single IF conversion

In this case, the filters are designed to allow the LSB to pass alone. Hence, increasing the phase of the signal at the RF decreases the phase at the IF. The signs of the phases ϕ_m , ϕ_n and the visibility phase ϕ_v will be opposite of the USB case. We accordingly modify the USB phase equations and rewrite them as:

$$\phi_m = 2\pi (\nu_{\text{LO}} - \nu) \tau_g + \theta_m$$

$$\phi_n = -2\pi \nu \tau_i + \theta_n$$

The correlator output r_{lsb} for the lower side-band can be expressed as:

$$r_{\text{lsb}} = \text{Re} \left[A_e |\mathcal{V}| e^{-j\{2\pi\nu_{\text{LO}}\tau_g + (\theta_m - \theta_n - \phi_v)\}} \int_{-\infty}^{\infty} H_m(\nu) H_n^*(\nu) e^{j2\pi\nu\Delta\tau} d\nu \right]$$

After introducing the instrumental gain G_{mn} , we rewrite the above equation as:

$$r_{\text{lsb}} = |\mathcal{V}| |G_{mn}(\Delta\tau)| \cos [2\pi (\nu_{\text{LO}} \tau_g - \nu_0 \Delta\tau) + (\theta_m - \theta_n) - \phi_v - \phi_G]$$

Cross-Corr.: LSB with multi IF conversions

It may sometimes be advantageous to up or down convert the signals more than once. These are done at several stages of the receiving system using different mixers and LO combinations. For a once down converted signal, the LSB spectrum gets flipped (frequencies at the lower end of RF appear at the higher end of IF). If the down conversion is done once again, the spectrum flips back in the second IF. Hence, if the number of down conversions are odd the final IF has a reversed spectrum.

The LSB correlator output $r_{\text{even}(lsb)}$ for the even number of IF conversions is:

$$r_{\text{even}(lsb)} = |\mathcal{V}| |G_{mn}(\Delta\tau)| \cos \left\{ 2\pi \left(\nu_{\Sigma LO} \tau_g + \nu_0 \Delta\tau \right) + \left(\theta_{\Sigma m} - \theta_{\Sigma n} \right) - \phi_v + \phi_G \right\}$$

Here, $\nu_{\Sigma LO}$ represents the sum of LO frequencies, $\theta_{\Sigma m}$ and $\theta_{\Sigma n}$ are respectively the combined phases resulting from the LOs for the paths m and n .

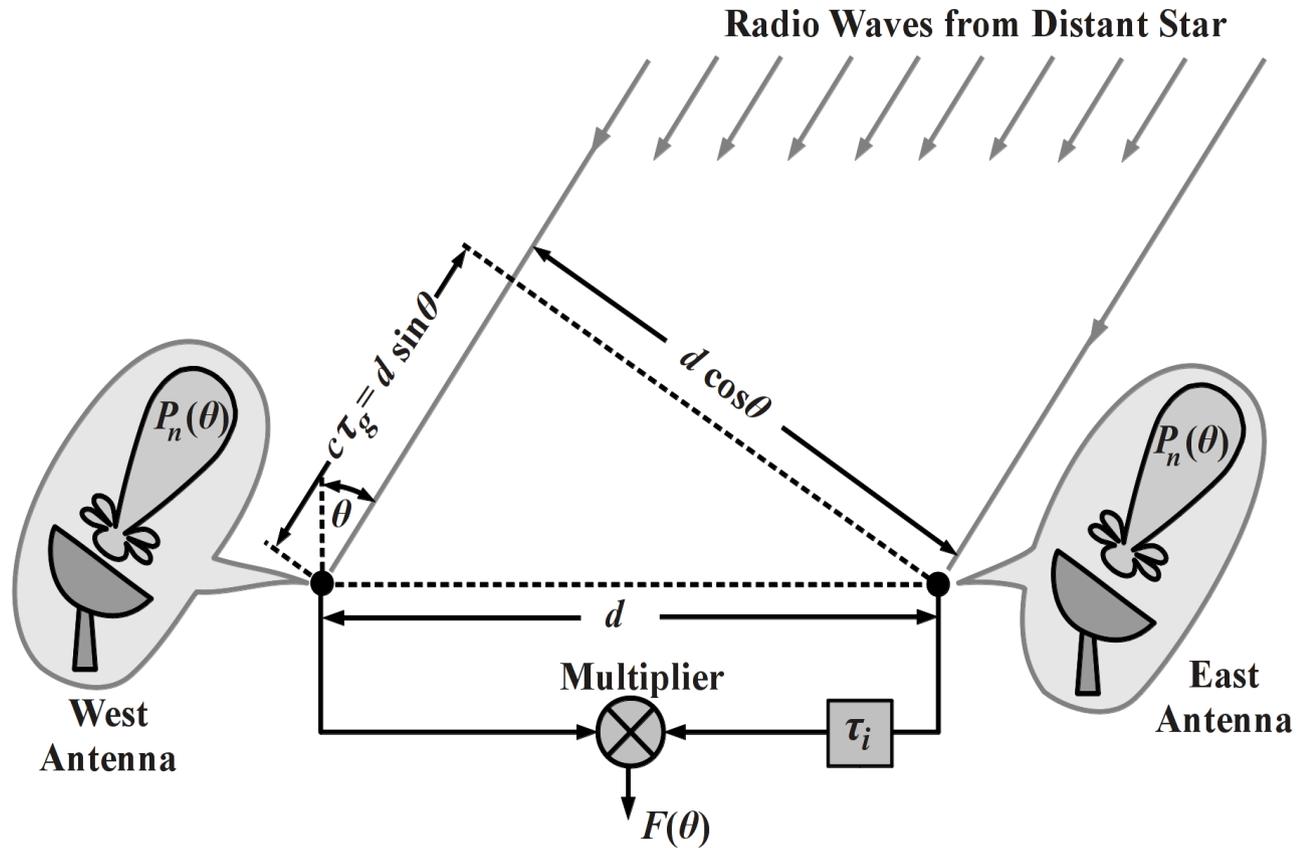
The LSB correlator output $r_{\text{odd}(lsb)}$ for odd number of IF conversions is:

$$r_{\text{odd}(lsb)} = |\mathcal{V}| |G_{mn}(\Delta\tau)| \cos \left\{ 2\pi \left(\nu_{\Sigma LO} \tau_g - \nu_0 \Delta\tau \right) + \left(\theta_{\Sigma m} - \theta_{\Sigma n} \right) - \phi_v - \phi_G \right\}$$

Here, $\nu_{\Sigma LO}$ represents the sum of LO frequencies, $\theta_{\Sigma m}$ and $\theta_{\Sigma n}$ are respectively the combined phases resulting from the LOs for the paths m and n .

Fringe Stopping in SSB Correlators

First we recall the basis of fringe stopping in an interferometer which does not do IF conversion. The antennas are tracking a radio source. Propagation delay $d \sin \theta$ keeps changing with Earth rotation. Compensate the geometrical delay $\tau_g = d \sin \theta / c$ by equal amount of instrumental delay τ_i such that $\Delta \tau = \tau_g - \tau_i = 0$.



Hence, signals entering the correlator are in phase. The interferometer will thus produce a peak output. This point of observation on the sky is called the *fringe stopping center*.

For a heterodyne system, we saw that correlator also produces a natural fringe frequency component ν_{fringe} . To stop these fringe oscillations, a continuous phase change is applied to one of the LO. How? See next.

Fringe Stopping in SSB Correlators

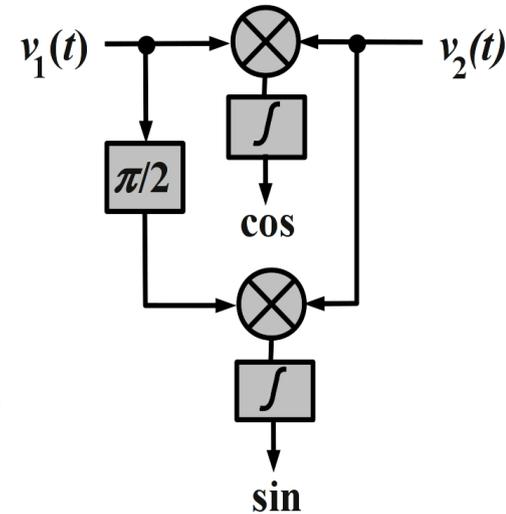
We know that the USB and LSB correlator outputs are given as:

$$r_{\text{usb}} = |\mathcal{V}| |G_{mn}(\Delta\tau)| \cos [2\pi (\nu_{\text{LO}} \tau_g + \nu_0 \Delta\tau) + (\theta_m - \theta_n) - \phi_v + \phi_G]$$

$$r_{\text{lsb}} = |\mathcal{V}| |G_{mn}(\Delta\tau)| \cos [2\pi (\nu_{\text{LO}} \tau_g - \nu_0 \Delta\tau) + (\theta_m - \theta_n) - \phi_v - \phi_G]$$

Fringe oscillations can be eliminated by varying $\theta_m - \theta_n$ at a rate which maintains a constant modulo 2π to the term $[2\pi\nu_{\text{LO}}\tau_g + (\theta_m - \theta_n)]$. This is achieved by either by adding $2\pi\nu_{\text{fringe}}$ to θ_m or subtracting it from θ_n .

Complex correlators are preferred in fringe stopping, since they produce real and imaginary parts. The cos output r_{real} and sin output r_{imag} are respectively real and imaginary.



Under fringe stop conditions:

$$r_{\text{real}} = \text{Re} [\mathcal{V}] \int_{-\infty}^{\infty} H_m(\nu) H_n^*(\nu) d\nu$$

$$r_{\text{imag}} = \text{Im} [\mathcal{V}] \int_{-\infty}^{\infty} H_m(\nu) H_n^*(\nu) d\nu$$

Mathematically, two conditions are maintained: (i) $\Delta\tau = 0$ and (ii) $2\pi\nu_{\text{LO}}\tau_i + (\theta_m - \theta_n) = 0$. Hence, correlator outputs r_{real} and r_{imag} respectively represent the real and imaginary parts of $G_{mn} \mathcal{V}(u, \nu)$.

Cross-Correlator Double Side Band Response

Right figure:

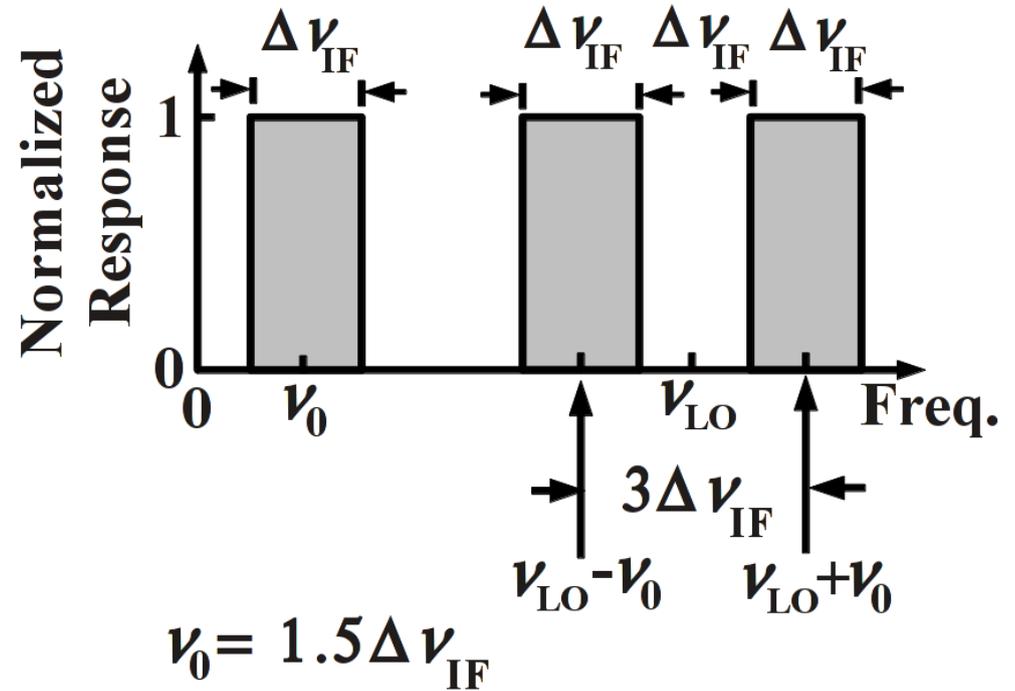
Spectra of the input and sidebands for $\nu_0 = 1.5\nu_{IF}$. The band-pass filters block everything except the two side-band signals. Hence the DSB complex correlator's cosine (real) response $r_{dsb(cos)}$ is the sum of r_{usb} and r_{lsb} (obtained earlier) is:

$$\begin{aligned} r_{dsb(cos)} &= r_{usb} + r_{lsb} \\ &= 2|\mathcal{V}| |G_{mn}(\Delta\tau)| \cos\{2\pi\nu_0\Delta\tau + \phi_G\} \cos\{2\pi\nu_{LO}\tau_g + (\theta_m - \theta_n) - \phi_v\} \end{aligned}$$

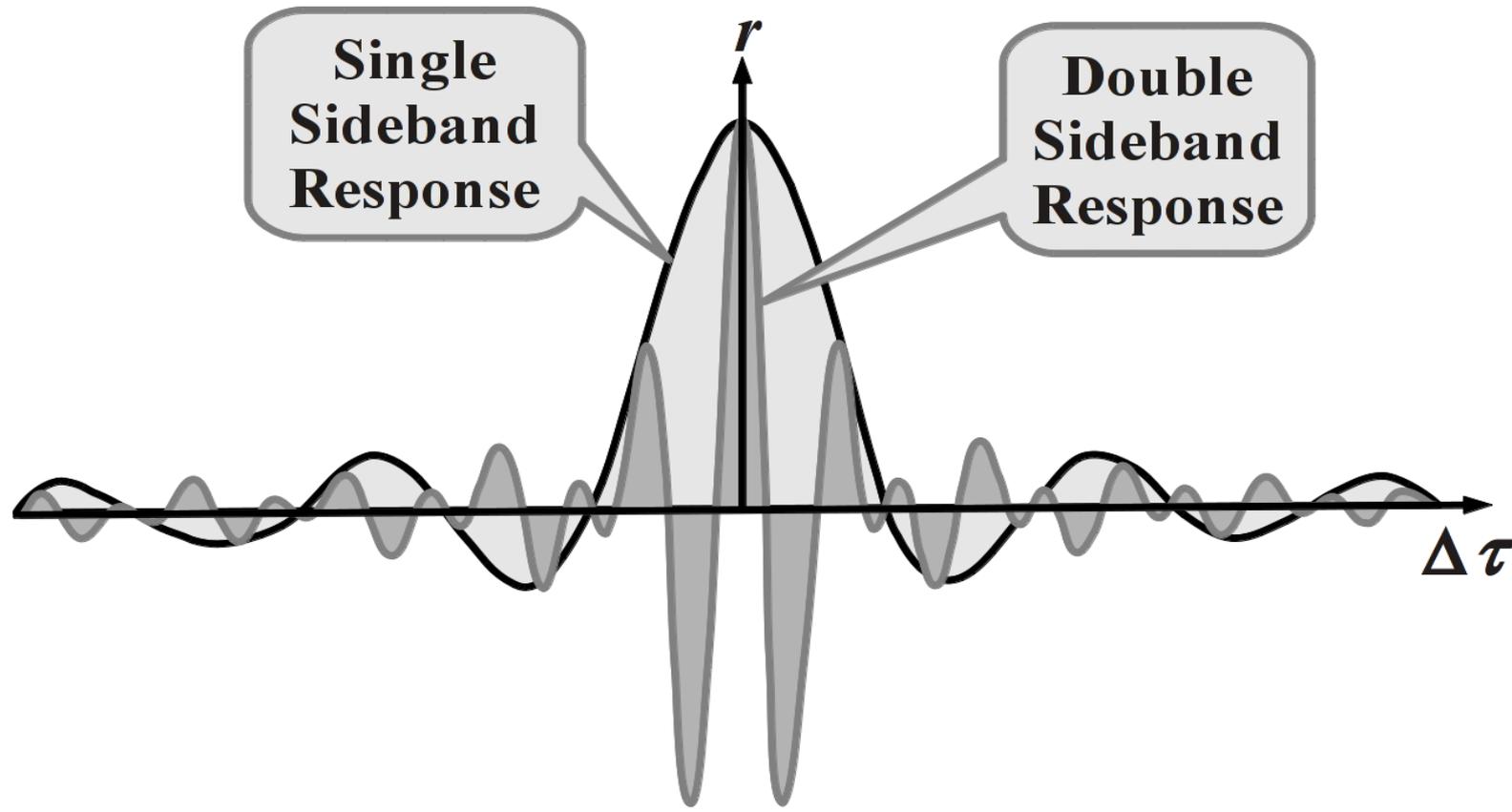
The sin (imaginary) output $r_{dsb(sin)}$ can be logically produced from above as:

$$r_{dsb(sin)} = 2|\mathcal{V}| |G_{mn}(\Delta\tau)| \sin\{2\pi\nu_0\Delta\tau + \phi_G\} \cos\{2\pi\nu_{LO}\tau_g + (\theta_m - \theta_n) - \phi_v\}$$

The last term $\cos\{2\pi\nu_{LO}\tau_g + (\theta_m - \theta_n) - \phi_v\}$ is independent of $\Delta\tau$ and ϕ_G . It rapidly modulates the fringe amplitude, which can be seen by stopping τ_i such that $\Delta\tau$ changes.

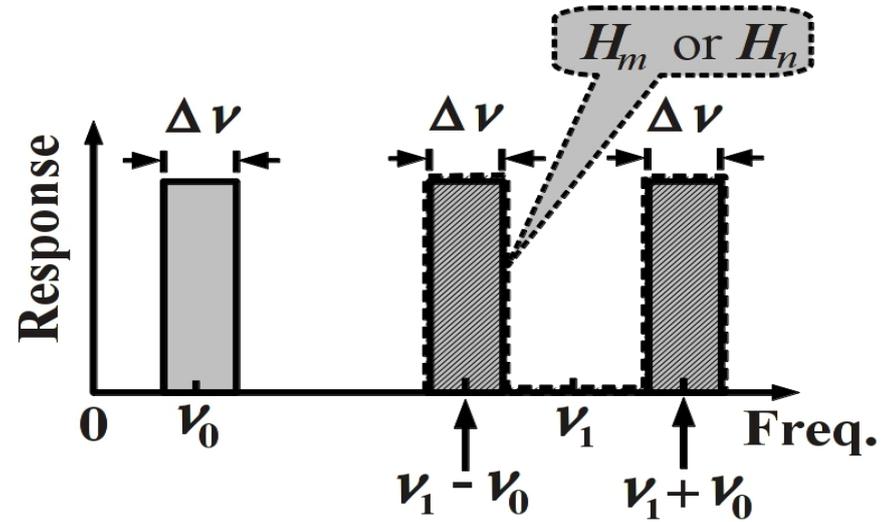
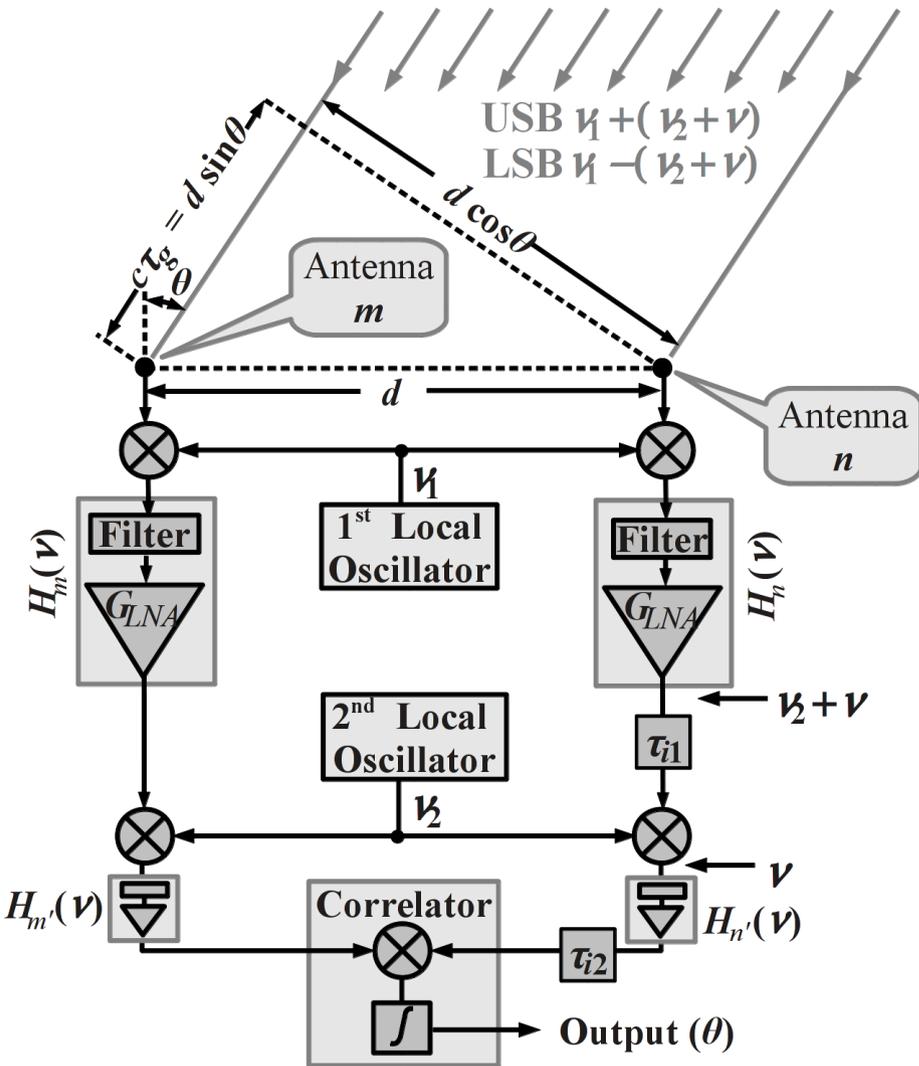


Cross-Correlator Double Side Band Response

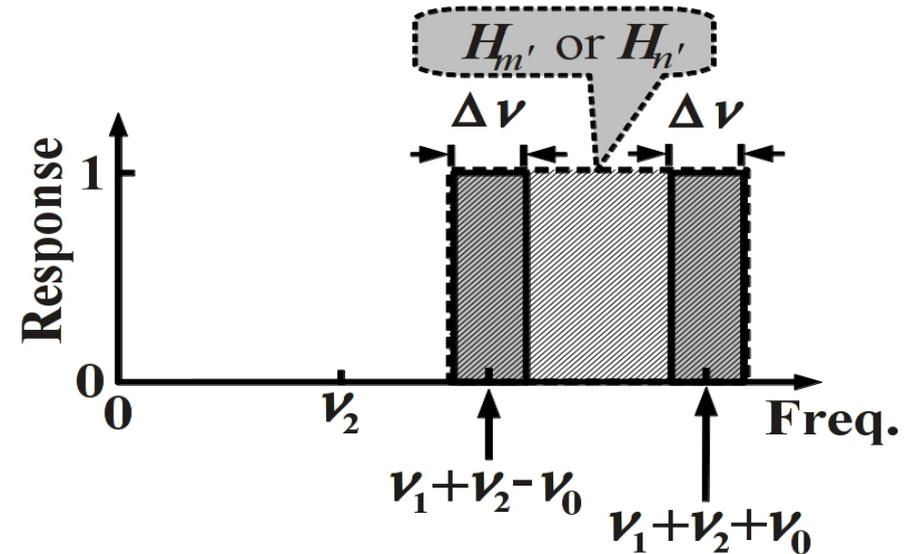


A tentative fringe amplitude variation as a function of $\Delta\tau$, where the center frequency ν_0 of the received signal is taken as $1.5\Delta\nu_{\text{IF}}$ (see previous diagram) and the instrumental delay τ_i is kept constant. If the term $2\pi\nu_0\Delta\tau + \varphi_G$ is adjusted to maximize the real output, the imaginary output vanishes and vice versa. Hence, *in continuum observations where both the side-bands are of equal strength, the complex correlator does not give any extra sensitivity.*

Correlator with DSB with multiple IFs



Response of H_m or H_n (first IF).



Response of $H_{m'}$ or $H_{n'}$ (second IF).

Interferometer with two IF stages. Next to the antennas DSB receivers are used. The second conversion has only USB. Two compensating delays (τ_{i1} and τ_{i2}) are used.

Correlator with DSB with multiple IFs

The signal phases ϕ_m and ϕ_n at the correlator input are:

$$\phi_m = \mp 2\pi (\nu_1 \pm \nu_2 \pm \nu) \tau_g \mp \theta_{m1} - \theta_{m2}$$

$$\phi_n = -2\pi \{(\nu_2 + \nu) \tau_{i1} + \nu \tau_{i2}\} \mp \theta_{m1} - \theta_{m2}$$

Note that upper signs in the symbols \pm and \mp represent the USB conversions at both first and second IF for each antenna. The lower signs represent the same for the lower side-band. The correlator responses are:

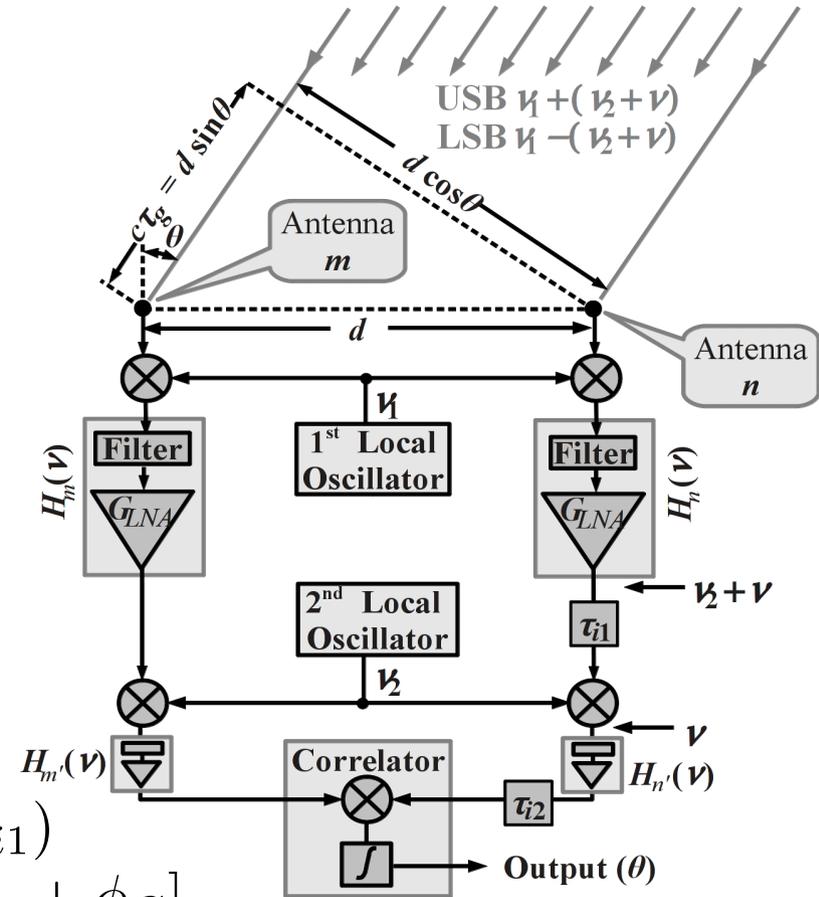
$$r_{\text{usb}} = |\mathcal{V}| |G_{mn}(\Delta\tau)| \cos [2\pi\nu_1\tau_g + 2\pi\nu_2(\tau_g - \tau_{i1}) + 2\pi\nu_0\Delta\tau + (\theta_{m1} - \theta_{n1}) + (\theta_{m2} - \theta_{n2}) - \phi_v + \phi_G]$$

$$r_{\text{lsb}} = |\mathcal{V}| |G_{mn}(\Delta\tau)| \cos [2\pi\nu_1\tau_g - 2\pi\nu_2(\tau_g - \tau_{i1}) - 2\pi\nu_0\Delta\tau + (\theta_{m1} - \theta_{n1}) - (\theta_{m2} - \theta_{n2}) - \phi_v - \phi_G]$$

The complete double side-band response is given as:

$$r_{\text{dsb}} = r_{\text{usb}} + r_{\text{lsb}} = 2|\mathcal{V}| |G_{mn}(\Delta\tau)| \cos [2\pi \{ \nu_2 (\tau_{i1} - \tau_g) - \nu_0 \Delta\tau - (\theta_{m2} - \theta_{n2}) - \phi_G \}] \cos \{ \nu_1 \tau_g + (\theta_{m1} - \theta_{n1}) - \phi_v \}$$

where, $\Delta\tau = \tau_g - \tau_{i1} - \tau_{i2}$.



Correlator with DSB with multiple IFs

The complete double side-band response was:

$$r_{\text{dsb}} = r_{\text{usb}} + r_{\text{lsb}} = 2|\mathcal{V}| |G_{mn}(\Delta\tau)| \cos [2\pi \{ \nu_2 (\tau_{i1} - \tau_g) - \nu_0 \Delta\tau \} - (\theta_{m2} - \theta_{n2}) - \phi_G] \cos \{ \nu_1 \tau_g + (\theta_{m1} - \theta_{n1}) - \phi_v \}$$

where, $\Delta\tau = \tau_g - \tau_{i1} - \tau_{i2}$.

First cosine term modulates the fringe amplitude. Fringe phase is determined by the second cosine term which depends only on phase LO_1 . Hence, the phase shift should be applied to LO_1 . The effects of τ_{i1} and τ_{i2} are seen when one is held constant and other is varied as explained:

(i) Let τ_{i1} be variable which compensates the delay and let $\tau_{i2} = 0$. Hence in the last equation, $\tau_{i1} - \tau_g$ must be ideally zero and ϕ_G should be minimized. It follows that θ_{m2} and θ_{n2} must be equalized to maximize the fringe amplitude.

(ii) Let τ_{i2} be variable which compensates the delay and let $\tau_{i1} = 0$. This is preferred in large arrays with digital delay compensation. With changing τ_g , a continuously varying phase shift is applied in θ_{m2} and θ_{n2} . This is to maintain the first cosine term near unity. The fringe phase is not affected but its amplitude varies.

Fringe Stopping in DSB Corrs. with Multi-IFs

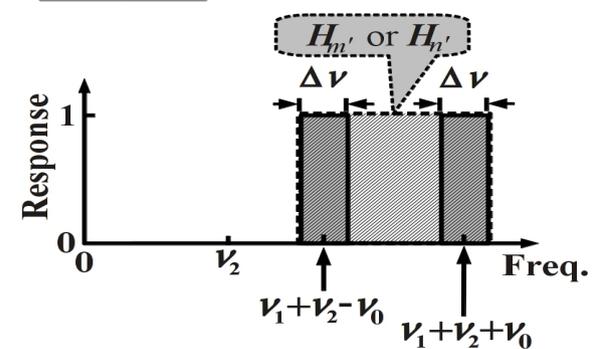
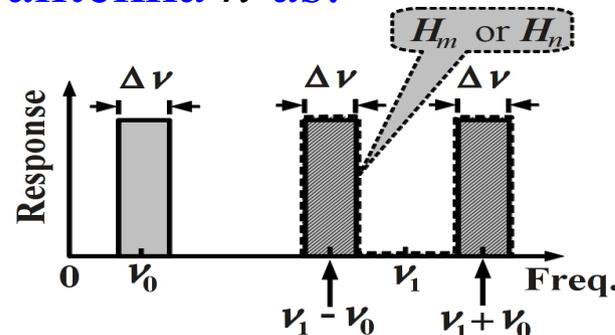
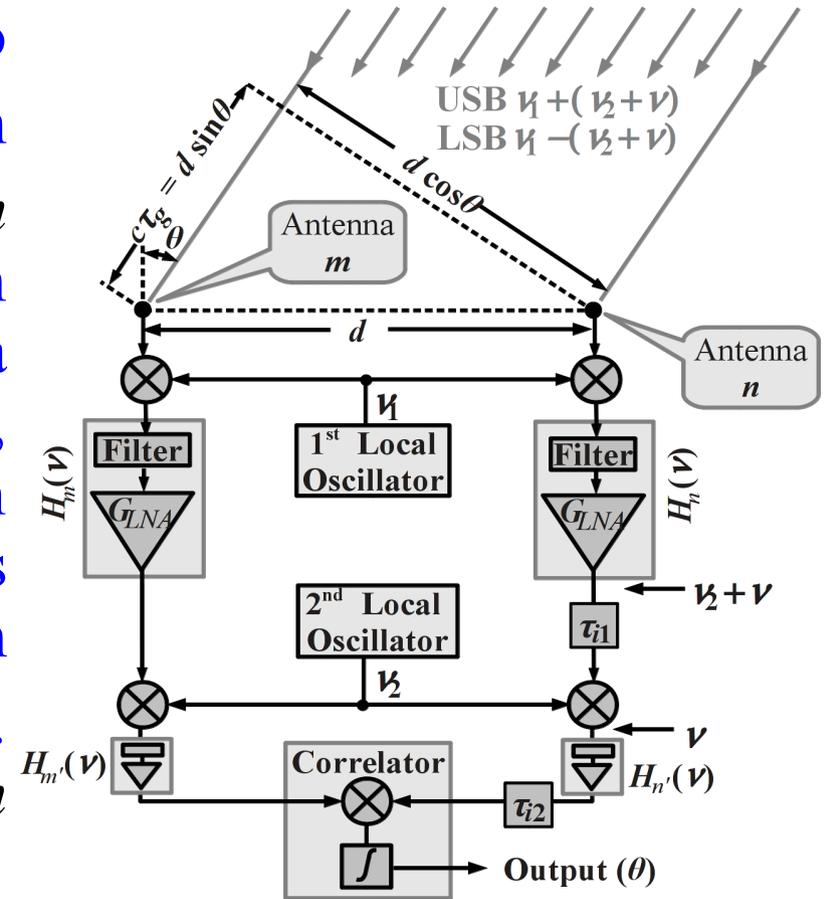
Let $\tau_{i1} = 0$ and let τ_{i2} be changed continuously to maintain $\Delta\tau_g = 0$. As the source moves, path difference between signals at antenna m and n changes resulting in fringe frequency. With respect to wave-front at antenna n , the antenna m moves away as a function of time. Thus, there is a relative Doppler shift occurring in antenna m with respect to antenna n . This results from changing geometrical delay τ_g with source position as a function of time. Mathematically, RF frequency ν_{mRF} at antenna m is related to RF frequency ν_{nRF} of antenna n as:

$$\nu_{mRF} = \nu_{nRF} \left(1 - \frac{d\tau_g}{dt} \right)$$

For USB signals from antenna m ,

the correlator input frequency is: $\nu_{mCRR(in)} = \nu_{nRF} \left(1 - \frac{d\tau_g}{dt} \right) + (\nu_1 + \nu_2)$

where ν_1 and ν_2 are the LO frequencies.



Fringe Stopping in DSB Corrs. with Multi-IFs

For the USB signals from antenna m , the correlator input frequency is :

$$\nu_{m\text{CRR}(\text{in})} = \nu_{n\text{RF}} \left(1 - \frac{d\tau_g}{dt} \right) + (\nu_1 + \nu_2) \dots (1)$$

If $\tau_{i1} = \tau_{i2} = 0$, corr. (USB) input freq. (Ant. n): $\nu_{n\text{CRR}(\text{in})} = \nu_{n\text{RF}} + (\nu_1 + \nu_2) \dots (2)$

For fringe stopping, correlator (USB) input freq. $\nu_{n\text{CRR}(\text{in})}$ from antenna n is:

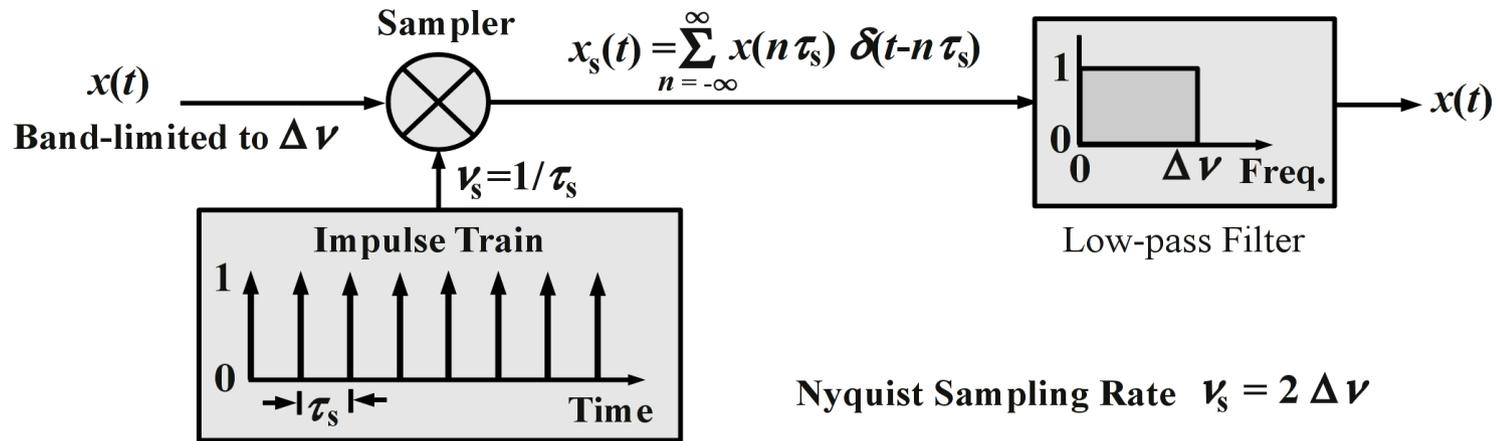
$$\nu_{n\text{CRR}(\text{in})} = \left\{ \nu_{n\text{RF}} + (\nu_1 + \nu_2) \left(1 + \frac{d\tau_g}{dt} \right) \right\} \left(1 - \frac{d\tau_g}{dt} \right) \dots (3)$$

For fringe stopping, $\nu_{n\text{RF}}$ is suitably decreased such that correlator have identical frequency inputs. For this, the two LO frequencies are multiplied by $(1+d\tau_g/dt)$. Mathematically, we add $2\pi(d\tau_g/dt)\nu_1$ to θ_{n1} and $2\pi(d\tau_g/dt)\nu_2$ to θ_{n2} . Simultaneously, the antenna n signal is delayed by τ_{i2} . Since the delay τ_{i2} is adjusted continuously, the signal frequency gets reduced by a factor $(1-d\tau_g/dt)$ which explains (3).

Note: If second order terms of $d\tau_g/dt$ in (3) are neglected, it reduces to (1). If signs of both $\nu_{n\text{RF}}$ and ν_1 are reversed, (1) and (3) can be applied on LSB, thus making both correlator input frequencies identical. The net result is fringes get stopped in both side-bands.

Digital Conversion: Base-Band Sampling

If power spectrum of a signal is restricted within a band of frequencies, the signal is said to be *band-limited*. If this band starts at zero frequency and ends at some fixed upper frequency (low-pass), it is known as *base-band*.



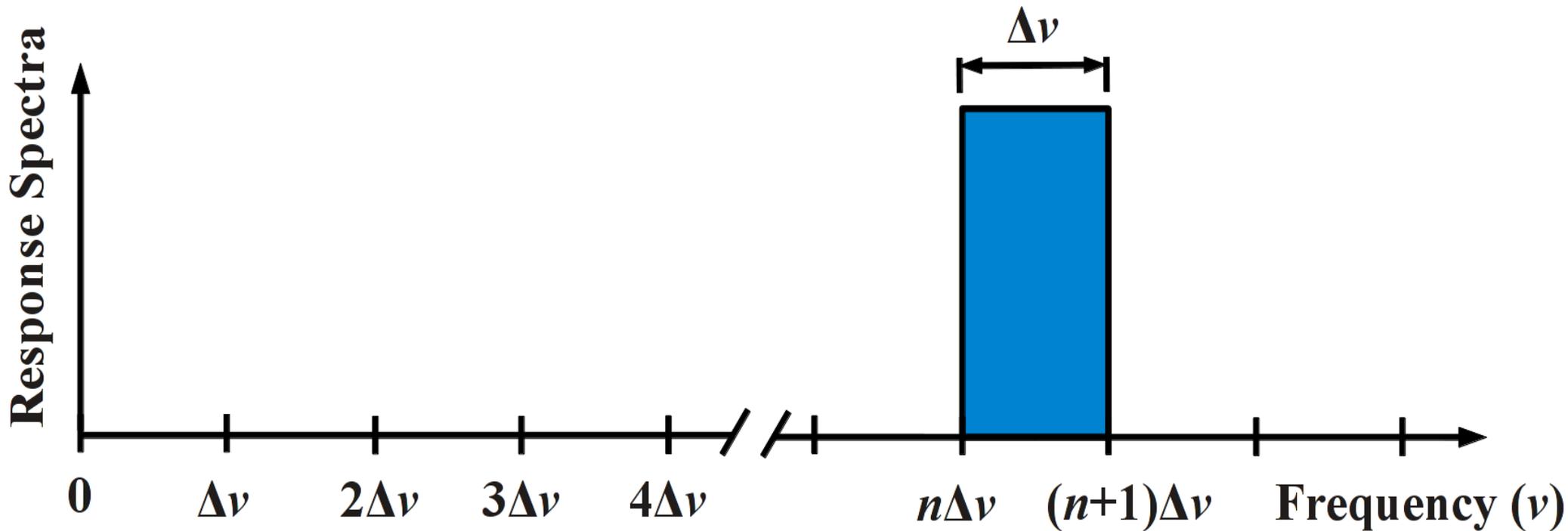
A base-band signal $x(t)$ having a maximum frequency component of $\Delta\nu$ (same as bandwidth) is sampled by multiplying it with an equally spaced train of pulses (delta functions). The separation between the pulses is τ_s which is known as *sampling period*. The resulting sampled signal $x_s(t)$ is given as:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n\tau_s) \delta(t - n\tau_s) \quad \text{where, } n \text{ is an integer.}$$

The smallest sampling rate at which the information content of the signal remains intact (after sampling) is known as *Nyquist rate* ν_s given as: $\nu_s = 2\Delta\nu$. The analog signal can be reproduced back from $x_s(t)$ by passing it through a low-pass filter having a cut-off frequency $\Delta\nu$.

Digital Conversion: Band-Pass Sampling

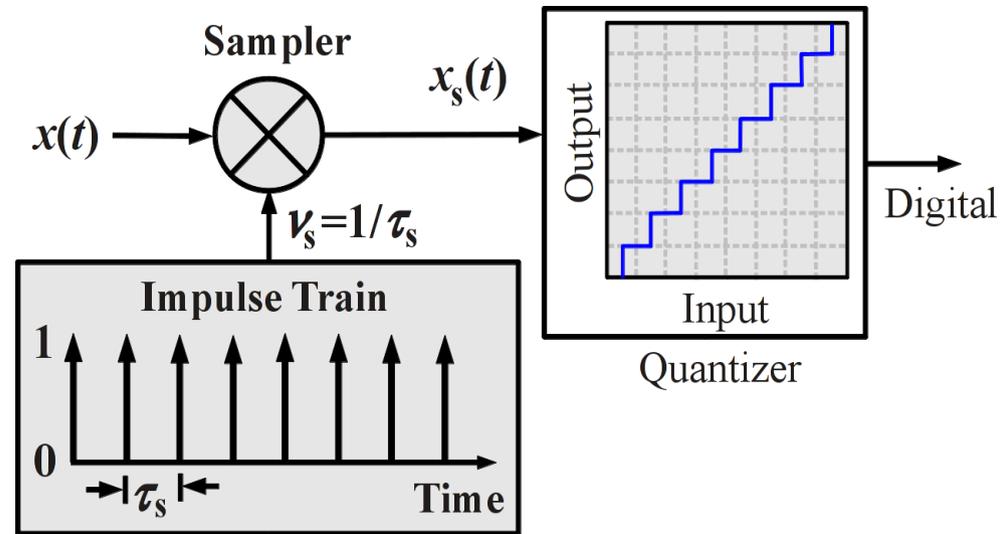
Band-pass spectra can also be sampled at Nyquist rate. Here, the analog signals lies within a frequency range starting from $n\Delta\nu$ to $(n+1)\Delta\nu$, where n is an integer. In other words, the signal spectrum starts at some integral multiple of $\Delta\nu$ instead of zero frequency. Since the signal band-width is $\Delta\nu$, the Nyquist rate remains same as $2\Delta\nu$.



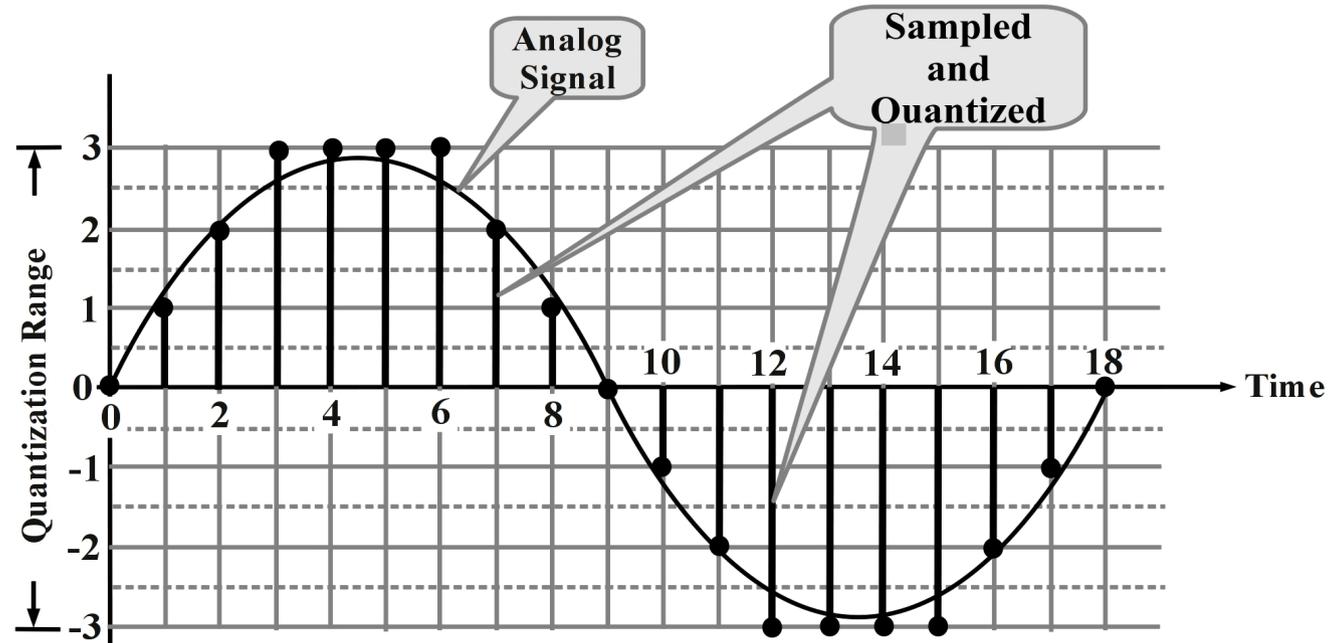
The necessary condition for band-pass sampling at Nyquist rate is that the lower and upper limits of the spectral band must be integral multiples of the band-width.

Digital Conversion: Quantization

A quantizer is used to convert the sampled outputs within a limited number of levels called *quantization levels*. It introduces an error known as *quantization error*. This error can be reduced by increasing the number of quantization levels. The quantized samples when represented in binary form are called *digital samples*.



Sampled values take the nearest quantization level. By increasing quantization levels, quantized values fall more closely to the signal. For infinite quantization levels, matching is exact.



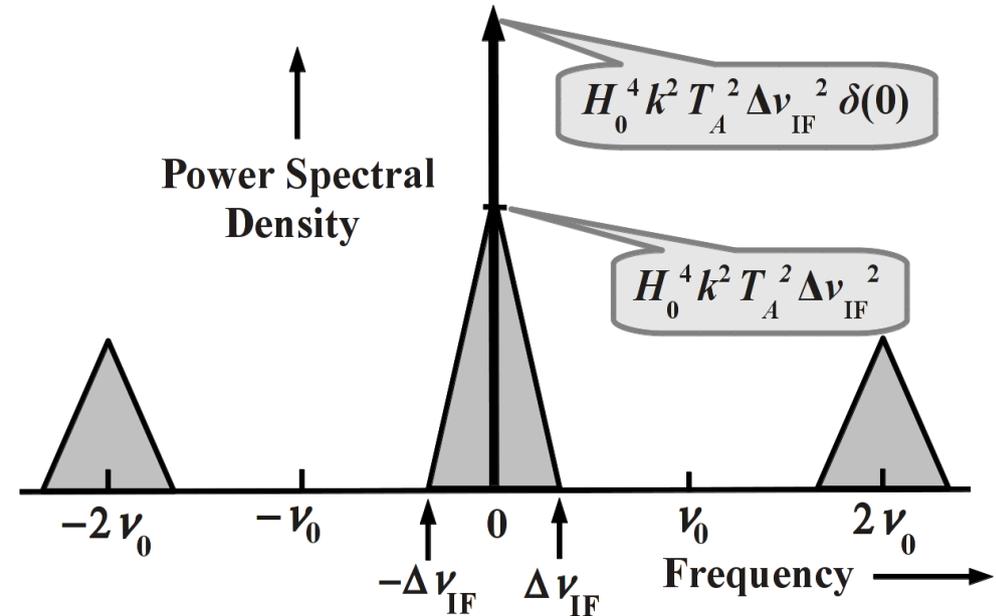
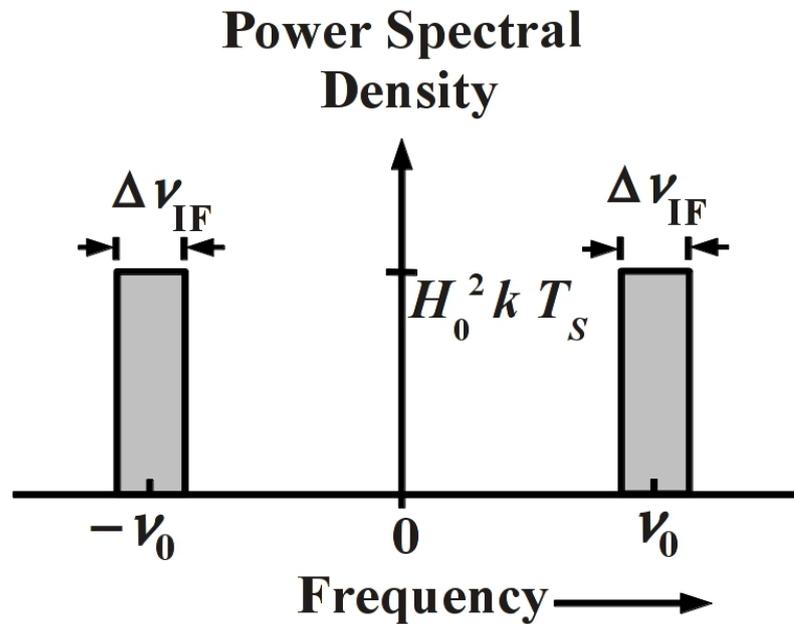
Cross Correlators: Signal to Noise Ratio

The SNR (signal to noise power ratio) is considered at the final output. Power contribution from the source alone is considered as signal. Power contribution from rest of the system is considered as noise. These contributions depend on:

- (i) Antenna temperature.
- (ii) System temperature.
- (iii) Receiver system band-width.
- (iv) Integration band-width after multiplication.

In the following discussions, we assume the source to be at the center of the fringe pattern. We may thus eliminate the effect of delay in our derivations. We also assume identical phase responses of the signal channels. The chosen correlator response is therefore the peak fringe amplitude, which is the modulus of visibility ($|\mathcal{V}|$).

Signal Spectra near Correlator



Input spectrum of correlator with rectangular pass-band $\Delta\nu_{IF}$. Antenna temperatures are assumed less than the respective system temperatures.

Complete spectrum from time-lag product (before integration) of two signals including noise bands. Only frequencies close to zero are able to pass out from the integrator. The required signal spectrum appears as a delta function at zero frequency (thick arrow).

$T_{Ant} \ll T_{Sys}$, k is Boltzmann constant. Since input spectra $|H_m(\nu)|^2$ and $|H_n(\nu)|^2$ contain both negative and positive frequencies and are symmetric about the origin, the output noise spectrum can be considered proportional either to (i) the convolution between $|H_m(\nu)|^2$ and $|H_n(\nu)|^2$, or to (ii) the cross-correlation between $|H_m(\nu)|^2$ and $|H_n(\nu)|^2$.

SNR in Analog Correlators

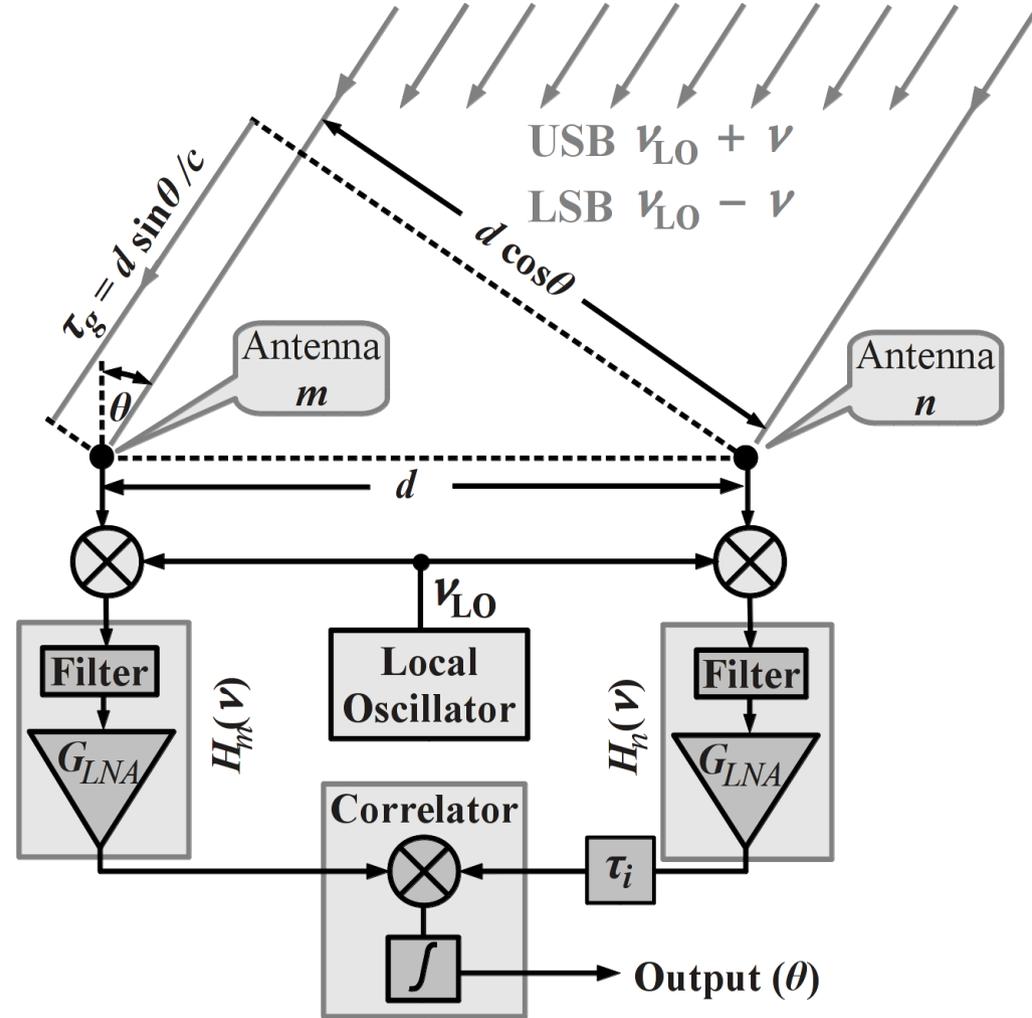
T_{Am} & T_{An} - Antenna temperatures of m and n respectively.

T_{Sm} & T_{Sn} - System temperatures respectively for paths associated with m and n antennas.

$H_m(\nu)$ & $H_n(\nu)$ - Respective responses of systems associated with m and n antennas.

$2\Delta\nu_{LF}$ - Equivalent integration bandwidth of correlator.

\mathcal{R}_{sn} - Signal to noise power ratio (SNR).



In general, $H_m(\nu)$ and $H_n(\nu)$ have identical rectangular band-widths $\Delta\nu_{IF}$. Unless the antenna temperatures are greater than their respective system temperatures, detection is not possible.

SNR in Analog Correlators

If $H_m(\nu)$ and $H_n(\nu)$ have identical rectangular band-widths $\Delta\nu_{IF}$, we obtain:

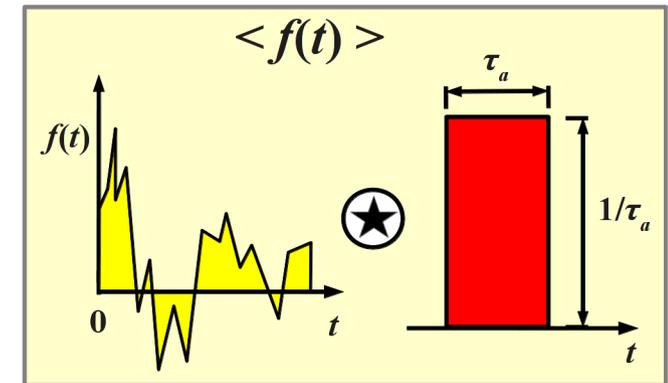
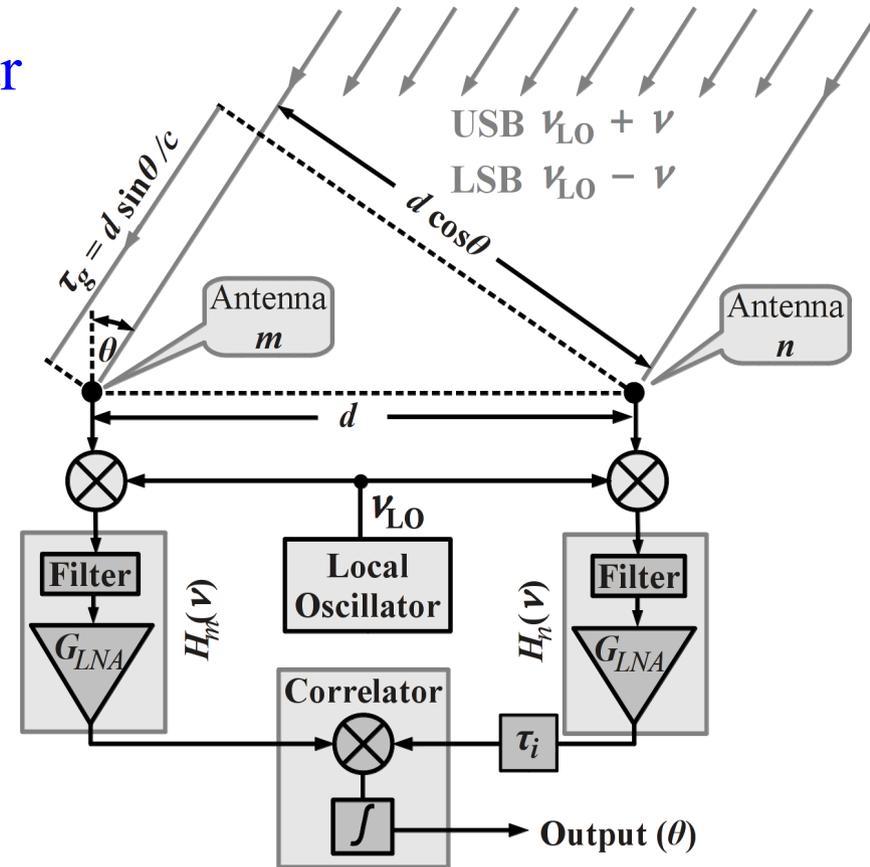
$$\mathcal{R}_{sn} = \sqrt{\frac{T_{Am}T_{An}}{T_{Sm}T_{Sn}}} \sqrt{\frac{\Delta\nu_{IF}}{\Delta\nu_{LF}}}$$

Let τ_a be the data averaging time of correlator. Averaging in time domain may be described as the convolution of the time signal with a rectangular function of unit area having a width τ_a . Hence the equivalent integration band-width $2\Delta\nu_{LF}$ (including both positive and negative frequencies) is related to τ_a as:

$$2\Delta\nu_{LF} = \int_{-\infty}^{\infty} \frac{\sin^2(\pi\tau_a\nu)}{(\pi\tau_a\nu)^2} d\nu = \frac{1}{\tau_a} \Rightarrow \Delta\nu_{LF} = \frac{1}{2\tau_a}$$

From above two equations,

the SNR (\mathcal{R}_{sn}) is given as:
$$\mathcal{R}_{sn} = \sqrt{2\Delta\nu_{IF}\tau_a \left(\frac{T_{Am}T_{An}}{T_{Sm}T_{Sn}} \right)}$$



SNR in Analog Correlators

$$\mathcal{R}_{\text{sn}} = \sqrt{2 \Delta\nu_{\text{IF}} \tau_a \left(\frac{T_{A_m} T_{A_n}}{T_{S_m} T_{S_n}} \right)}$$

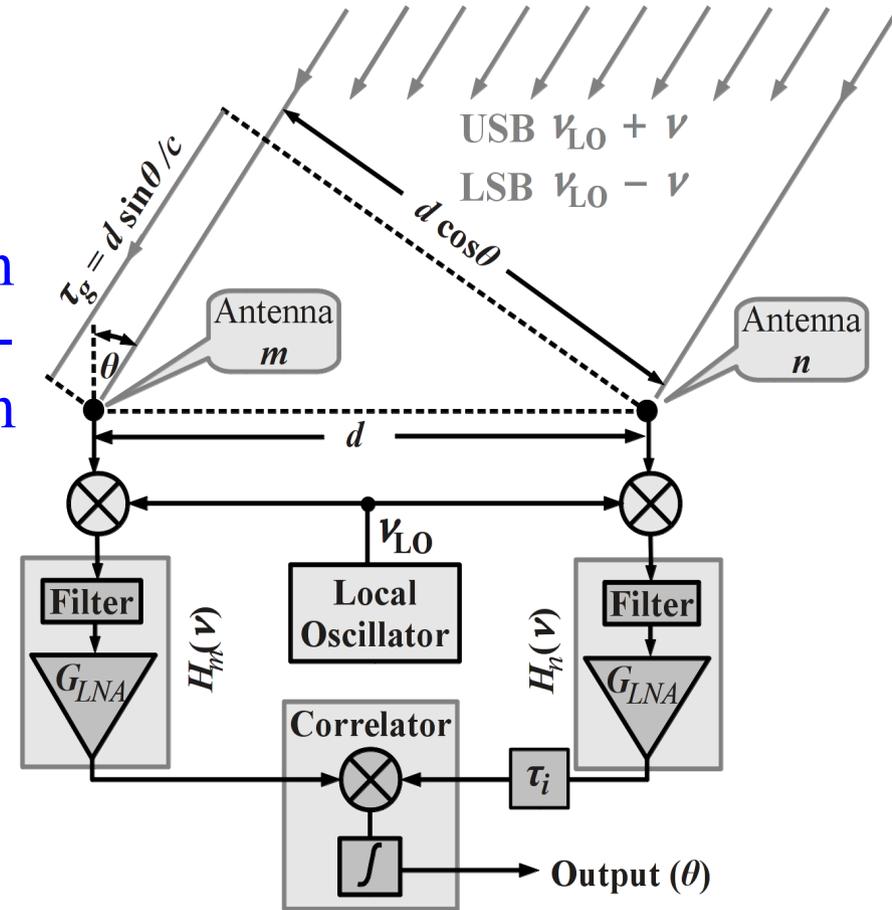
If the antennas are single polarized, then each antenna receives only half the total flux-density S . Then the received power p_r for each antenna is given as:

$$p_r = k T_A = \frac{1}{2} A_e S \Rightarrow T_A = \frac{1}{2} \frac{A_e S}{k}$$

where, A_e is antenna effective aperture area.

Let the antennas are identical in nature ($T_{A_m} = T_{A_n} = T_A$). The systems are also considered identical ($T_{S_m} = T_{S_n} = T_S$). Substituting T_A in the first equation, \mathcal{R}_{sn} is expressed as:

$$\mathcal{R}_{\text{sn}} = \frac{A_e S}{k T_S} \sqrt{\frac{\Delta\nu_{\text{IF}} \tau_a}{2}}$$



Note: SNR can be improved by increasing the factor $\sqrt{\Delta\nu_{\text{IF}} \tau_a}$

Thus, larger band-widths ($\Delta\nu_{\text{IF}}$) and large integration time (τ_a) improves the sensitivity by improving SNR.

SNR in Digital Correlators

The quantization process in digitizing the signals adds a little more noise to the system known as *quantization noise*. To include the quantization loss, an efficiency factor η_Q has been introduced. The final expression for SNR (\mathcal{R}_{sn}) is given as:

$$\mathcal{R}_{\text{sn}} = \frac{A_e S \eta_Q}{kT_S} \sqrt{\frac{\Delta\nu_{\text{IF}} \tau_a}{2}} \quad \text{where, } 0 \leq \eta_Q \leq 1.$$

In terms of antenna temperature the SNR is given as:

$$\mathcal{R}_{\text{sn}} = \frac{T_A \eta_Q}{T_S} \sqrt{2 \Delta\nu_{\text{IF}} \tau_a}$$

Note that in deriving the above expressions, no delay was introduced between the two signals reaching the correlator. Also the phase responses of the signal channels are assumed identical. Hence the source must be in the center of the fringe pattern, and the response is for the peak fringe amplitude.

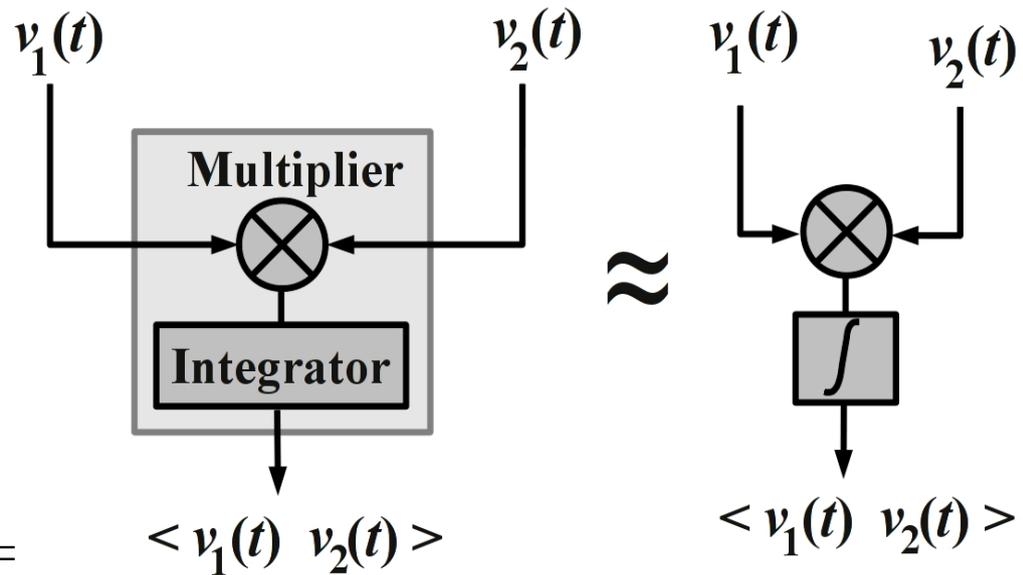
RMS noise at Complex Correlator Output

Consider a simple digital correlator whose SNR is:

$$\mathcal{R}_{\text{sn}} = \frac{A_e S \eta_Q}{k T_S} \sqrt{\frac{\Delta \nu_{\text{IF}} \tau_a}{2}}$$

We substitute $\mathcal{R}_{\text{sn}} = 1$ and replace S by S_{crr} . We may express S_{crr} as:

$$S_{\text{crr}} = \frac{\sqrt{2} k T_S}{A_e \eta_Q} \frac{1}{\sqrt{\Delta \nu_{\text{IF}} \tau_a}}$$

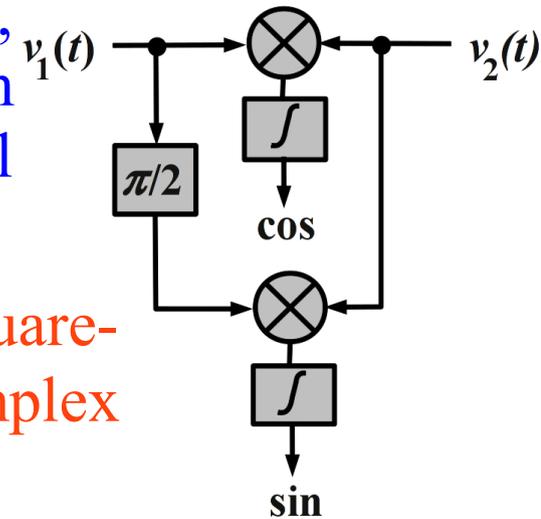


We may visualize S_{crr} as the flux-density (in watt/m²/Hz) of an unresolved source located at the phase reference point, that gives a peak fringe amplitude equal to the noise from the system. Here, T_S represents system temperature in K, k is Boltzmann constant, $\Delta \nu_{\text{IF}}$ represents the effective rectangular IF bandwidth, τ_a is data averaging time, A_e is the effective aperture area of each antenna, and η_Q is the efficiency factor relating quantization loss.

Note: Any unresolved radio source on phase reference center producing a flux density less than S_{crr} is not detectable at the correlator output.

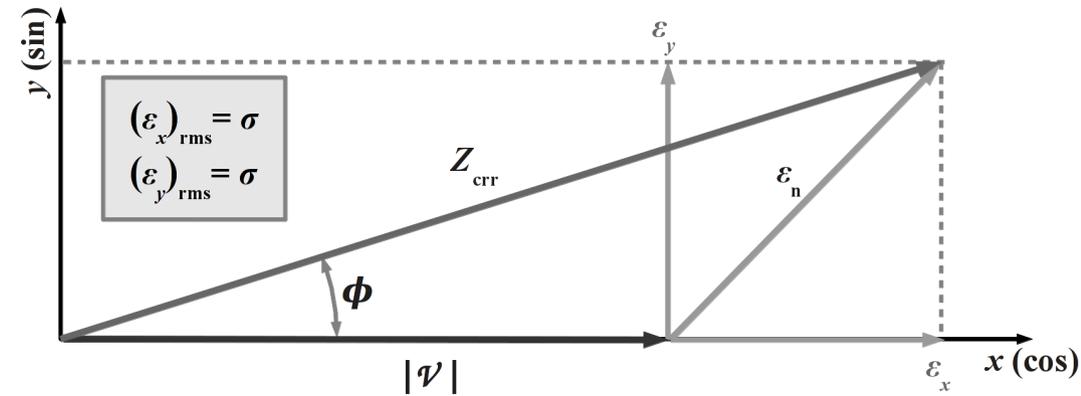
RMS noise at Complex Correlator Output

In a complex digital correlator under fringe stop condition, the output \cos (real) and \sin (imaginary) components form Hilbert transform pairs. However, the noise outputs (real and imaginary) are uncorrelated.



Let $|\mathcal{V}|$ be the modulus of noise-free visibility. It is the square-root of sum of squares of \cos and \sin outputs of a complex correlator. Let ε_n be the noise.

The correlator output Z_{crr} contain both the signal \mathcal{V} and noise ε_n . Using a vector diagram, with $|\mathcal{V}|$ as reference, we visualize the effect of noise on the correlator output.



The phase deviation caused in correlator output Z_{crr} by the noise is ϕ . The real and imaginary components of ε_n are represented by ε_x and ε_y respectively. The corresponding rms values are represented by $(\varepsilon_x)_{\text{rms}}$ and imaginary $(\varepsilon_y)_{\text{rms}}$ respectively. Each of these is equal to S_{crr} which is the uncertainty in the measurement. The noise ε_n has an rms amplitude $\varepsilon_{\text{rms}} = \sqrt{2} S_{\text{crr}}$.

RMS noise at Complex Correlator Output

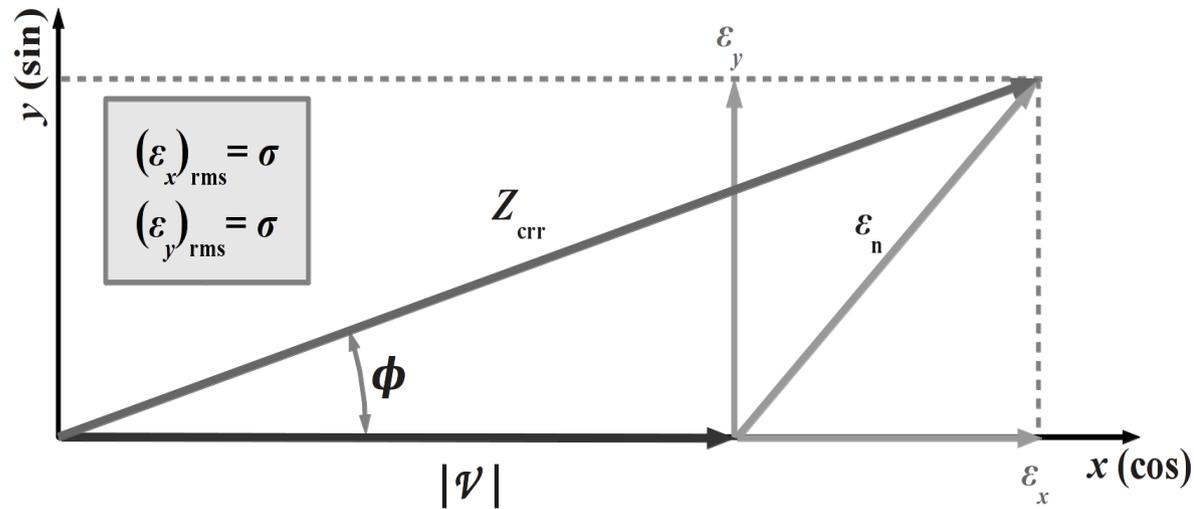
Phase deviation in Z_{crr} due to noise: ϕ

Real and imaginary parts of ϵ_n :

Real: ϵ_x , Imaginary: ϵ_y

Their rms values are:

Real: $(\epsilon_x)_{\text{rms}}$, Imaginary: $(\epsilon_y)_{\text{rms}}$



Both $(\epsilon_x)_{\text{rms}}$ and $(\epsilon_y)_{\text{rms}}$ equals S_{crr} which is the uncertainty in the measurement.

The rms amplitude of ϵ_n is $\epsilon_{\text{rms}} = \sqrt{2} S_{\text{crr}}$. The derivation is given below.

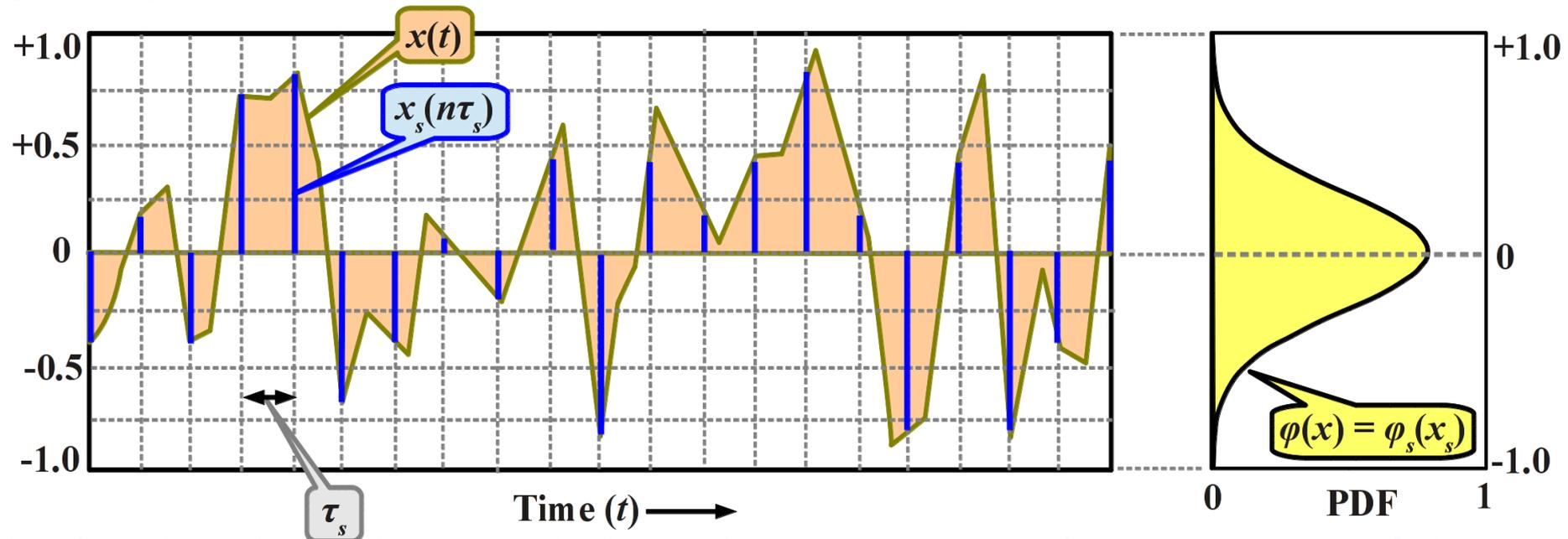
$$\epsilon_{\text{rms}} = \sqrt{(Z_{\text{crr}} \cdot Z_{\text{crr}}) - (Z_{\text{crr}})^2} = \sqrt{(\epsilon_n \cdot \epsilon_n)} = \sqrt{2} S_{\text{crr}}$$

When antennas are pointed to blank sky, the correlator output is due to noise alone which is measurable. By pointing antennas at a unresolved source, the correlator output Z_{crr} obtained is sum of visibility and noise. Measurement of Z_{crr} must be under fringe stop condition so that correlator output is maximum.

Important note: Source must be unresolved and both real and imaginary outputs of the complex correlator are required for computing Z_{crr} .

Basics of Gausssian probability distribution

A random signal $x(t)$ having a Gaussian probability distruibution and its sampled version $x_s(t)$ as a function of time are shown along with their respective probability density functions $\varphi(x)$ and $\varphi_s(x_s)$ taken over a large amount of time.



We study the signal and its sampled version over a very large amount of time.

Mean μ : Time average of a signal is known as its *mean*. The mean tends to zero if taken over a large time duration.

$$\text{Mean of } x(t) \text{ is: } \mu = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt$$

$$\text{Mean of } x_s(n \tau_s) \text{ is: } \mu = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} x_i$$

Basics of Gausssian probability distribution

Standard deviation σ : The Standard Deviation is a measure of how spread out numbers are from its mean value. To find standard deviation, one needs to calculate the variance σ^2 first.

Variance σ^2 : The square of standard deviation is called variance. They are given as:

$$\text{Variance of } x(t) \text{ is: } \sigma^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x(t) - \mu)^2 dt$$

$$\text{Variance of } x_s(n \tau_s) \text{ is: } \sigma^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2$$

Mean square value msv: The mean square values are given as:

$$\text{Mean square value of } x(t) \text{ is: } \text{msv} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (x(t))^2 dt$$

$$\text{Mean square value of } x_s(n \tau_s) \text{ is: } \text{msv} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (x_i)^2$$

The square-root of mean square value (msv) is know as the root mean square value (rms).

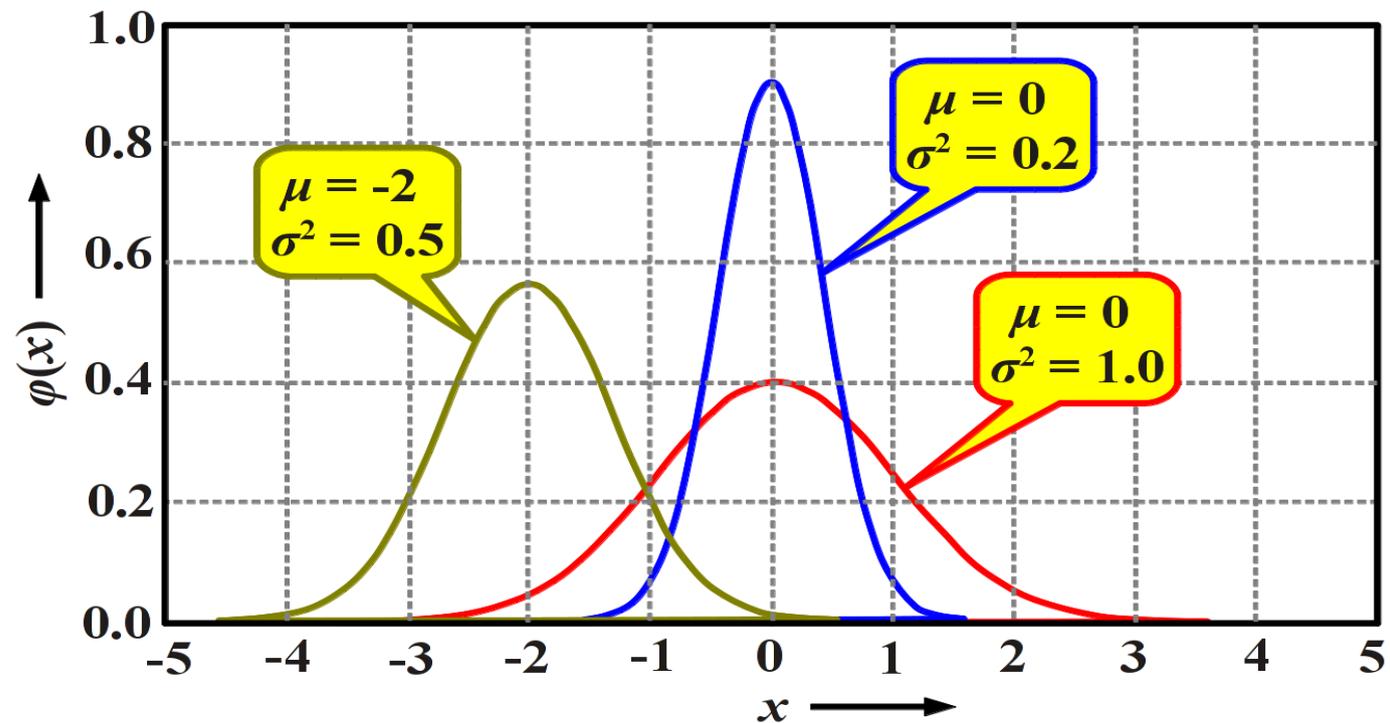
Note: If mean (μ) is zero, the variance (σ^2) equals mean square value (msv).

Basics of Gausssian probability distribution

Probability denisty function (PDF): The probability density function (PDF) is a mathematical function which describes the probability of occurrence of any or all possible events.

The PDF φ of a continuous random variable x having a Gaussian distribution (normal distribution) with mean μ and variance σ^2 is:

$$\varphi = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



Note:

- (i) For continuous random variable, area under the curve is unity since it represents the total probability of all events.
- (ii) For discrete random variable, the sum all probabilities (all events) is unity.
- (iii) The possible values of sampled signals (un-quantized) in our case of study are infinite and are assumed to be Gaussian.

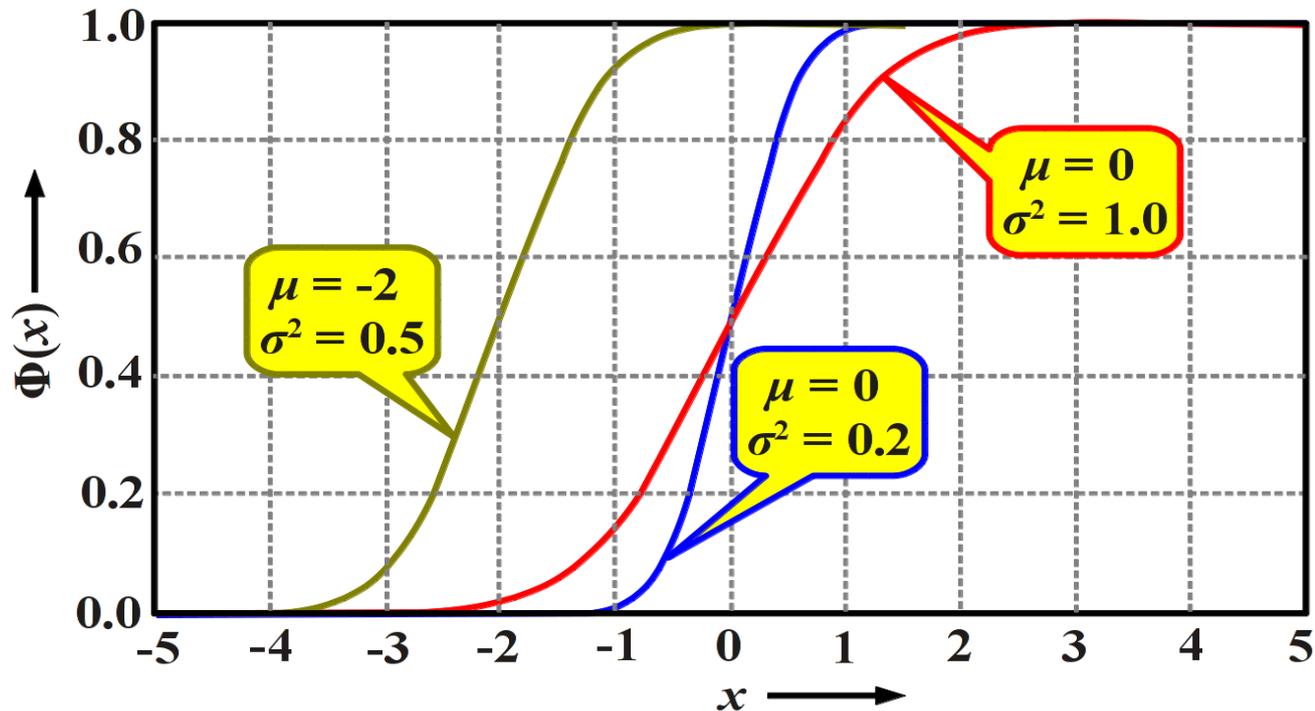
Basics of Gausssian probability distribution

Cumulative density function (CDF):

The cumulative density function (CDF) Φ describes the probability of a continuous random variable falling in the interval $(-\infty, x_0)$, where x_0 is an event of the random variable x .

For a normal distribution, it is computed as an integral of the PDF within the limits $(-\infty, x_0)$. For a contunuous random variable, it is evaluated as:

$$\Phi = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$



Note:

- (i) For continuous random variable CDF is called ***cumulative density function***, whereas, for discrete random variable it is called ***cumulative distribution function***.
- (ii) The possible values of sampled signals (un-quantized) in our case of study are infinite and are assumed to be Gaussian.

Basics of Gausssian probability distribution

Calculating probabilities using PDF:

To calculate the probability Φ_v that a continuous random variable takes such that its values lies between v_1 and v_2 we integrate the PDF by settings the limits of integration as v_1 and v_2 as:
$$\Phi_v = \frac{1}{\sigma \sqrt{2\pi}} \int_{v_1}^{v_2} e^{-\frac{1}{2} \left(\frac{x}{\sigma}\right)^2} dx$$

Error function erf(x):

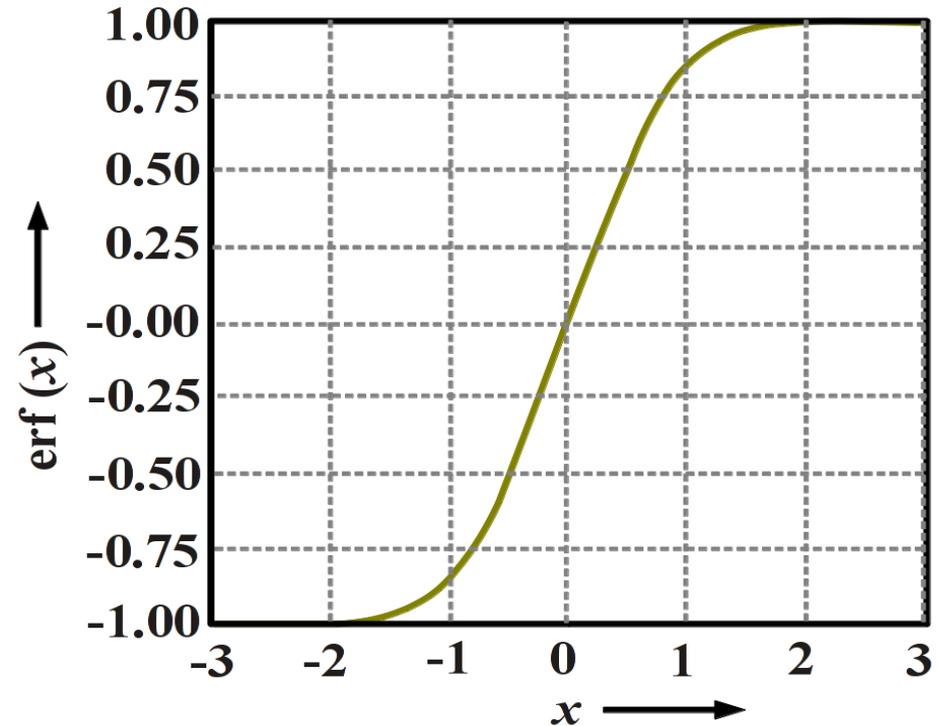
The error function erf(x) is twice the integral of the Gaussian distribution with 0 mean and variance of 1/2.

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

The error function is related to the cumulative density function Φ as:

$$\Phi(x) = \frac{1}{2} + \frac{1}{2} \text{erf} \left(\frac{x}{\sqrt{2}} \right)$$

The error function, evaluated at $x/(\sigma\sqrt{2})$ for positive x values, gives the probability that a measurement under the influence of normally distributed errors with standard deviation σ , has a distance less than x from the mean value.



Basics of Gausssian probability distribution

Product mean of two random variables: Two random variables are multiplied and averaged over a large time duration. A simple example is a digital correlator.

Product mean of two continuous random variables $x(t)$ and $y(t)$ is:

$$\langle x(t) y(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) y(t) dt$$

Product mean of two discrete random samples $x_s(n \tau_s)$ and $y_s(n \tau_s)$ is:

$$\langle x_s y_s \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N x_i y_i$$

Note: The product of two random numbers is also a random number. *Thus over a very large sample space, the product mean tends to zero provided the probability distribution is Gaussian with zero mean.*

Re-expressing variance: For a discrete random variable, the varaince σ^2 can also be expressed in another form as shown below:

$$\begin{aligned} \sigma^2 &= \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2 = \frac{1}{N} \sum_{i=1}^N x_i^2 - 2\mu \left(\frac{1}{N} \sum_{i=1}^N x_i \right) + \frac{1}{N} \sum_{i=1}^N \mu^2 \\ &= \langle x_s^2 \rangle - 2\mu(\mu) + \mu^2 = \langle x_s^2 \rangle - \mu^2 = \langle x_s^2 \rangle - \langle x_s \rangle^2 \end{aligned}$$

Thus we see

$$\sigma^2 = \langle x_s^2 \rangle - \langle x_s \rangle^2$$

Digital Correlator Systems

Though digital systems have technological limitations in context of high speed sampling, they are advantageous to analog systems in many respects. They offer better delay control and give highly accurate timing pulses to the system. For these reasons, most of the digital systems of today stand on cutting edge technology.

The analog signals are sampled at Nyquist rate (or higher) and converted to discrete form without any information loss. These discrete values are then approximated to a nearest available digital value through a process called *quantization*. The number of quantized states N_{qnt} comes from the number of bits N_{bits} used for representing a discrete value. These are related as $N_{\text{qnt}} = 2^{N_{\text{bits}}}$. Since N_{qnt} is a finite quantity, a distortion occurs in the digitized signal known as *quantization noise*. Hence, the digital correlator output is an approximation of the linear correlation functions $r(\tau)$ and $R(\tau)$.

Digital Correlators: Base-Band Sampling

Sampling at rates higher or lower than the Nyquist rate are respectively known as *over* and *under* samplings. Majority of radio telescope receivers convert its final IF into base-band, which is then sampled at slightly higher than the Nyquist rate and converted to digital after quantization.

Recall Wiener-Khinchin relationship: *The autocorrelation function of a wide-sense-stationary random process has a spectral decomposition given by the power spectrum of that process.* Thus for the rectangular pass-band, the auto-correlation function $R_{\infty}(\tau)$ as a function of time interval τ is:

$$R_{\infty}(\tau) = \frac{\sin(2\pi \Delta\nu \tau)}{2\pi \Delta\nu \tau}$$

Notes:

- (i) The sinc function is Fourier transform of the rectangular pass-band. Recall that Fourier transform of rectangular function is sinc function.
- (ii) The subscript symbol ∞ indicates infinite quantization levels (or state before quantization).

Digital Correlators: Band-Pass Sampling

Based on the Wiener-Khinchin relation, the auto-correlation $R_{\infty}(\tau)$ for a band-pass sampling is given as:

$$R_{\infty}(\tau) = \frac{\sin(2\pi \Delta\nu \tau)}{2\pi \Delta\nu \tau} \cos \left\{ 2\pi \left(n + \frac{1}{2} \right) \Delta\nu \tau \right\}$$

Notes:

- (i) $R_{\infty}(\tau)$ is zero when the time interval τ becomes an integral multiple of $1/(2\Delta\nu)$.
- (ii) For a rectangular shaped pass-band, the successive samples obtained at Nyquist rate are uncorrelated.
- (iii) The central frequency of the signal spectrum may be adjusted suitably for conforming it to a band-pass sampling. This will also minimize the sampling rate without any information loss. If the spectrum fails to conform, a slightly greater hypothetical band-width may be selected. The center frequency of this hypothetical band should be adjusted for proper band-pass sampling. The sampling will now be at slightly higher Nyquist rate (defined by the new hypothetical band-width).

Corr: Nyquist Sampled without Quantization

$x(t)$ & $y(t)$ - Band-limited signals.

$x_s(t)$ & $y_s(t)$ - Their sampled versions (correlator inputs).

Assumptions: (i) Sampled at Nyquist rates. (ii) Quantization is not done. (iii) Signals are in phase (time delay between them is zero).

Normalized correlation coefficient $r_{\text{norm}} = \frac{\langle x_s(t) y_s(t) \rangle}{\sqrt{\langle [x_s(t)]^2 \rangle \langle [y_s(t)]^2 \rangle}}$

$\langle x_s(t) y_s(t) \rangle$ is the expected or mean value of a large number of samples. This is like integrating $x(t) y(t)$ over a large amount of time. Assuming $x(t)$ and $y(t)$ possess equal variance (σ^2), we express the numerator of above equation as:

$$\langle x_s(t) y_s(t) \rangle = \langle x(t) y(t) \rangle = r_{\text{norm}} \sigma^2$$

Note: $\langle x(t) y(t) \rangle$ is the analog correlation between $x(t)$ and $y(t)$. The correlator output $r_{\infty}(\tau)$ (of unquantized signals $x_s(t)$ and $y_s(t)$) is effectively $\langle x(t) y(t) \rangle$.

x_i & y_i - The i^{th} samples in x_s and y_s .

N_{Nq} - Number of samples averaged.

The digital correlator output is: $r_{\infty} = \frac{1}{N_{\text{Nq}}} \sum_{i=1}^{N_{\text{Nq}}} x_i y_i$

Corr: Nyquist Sampled without Quantization

We obtained r_∞ only for a fixed no. of samples N_{Nq} . To represent the analog function $\langle x_s(t) y_s(t) \rangle$ more accurately, increase blocks of r_∞ . Since the numbers of x_i and y_i are large, they have Gaussian statistical nature as of $x(t)$ and $y(t)$.

Thus, $r_\infty = \frac{1}{N_{Nq}} \sum_{i=1}^{N_{Nq}} x_i y_i$ results $\langle r_\infty \rangle = r_{nrm} \sigma^2$ since $\langle r_\infty \rangle = \langle x(t) y(t) \rangle$

Also, $\langle r_\infty^2 \rangle = \frac{1}{N_{Nq}^2} \sum_{i=1}^{N_{Nq}} \sum_{k=1}^{N_{Nq}} \langle x_i y_i x_k y_k \rangle = \frac{1}{N_{Nq}^2} \left[\sum_{i=1}^{N_{Nq}} \langle x_i y_i \rangle^2 + \sum_{i=1}^{N_{Nq}} \sum_{k \neq i} \langle x_i y_i x_k y_k \rangle \right]$

Thus, a digital correlator output having infinite quantization levels is equivalent to a linear measure of the normalized cross-correlation r_{nrm} .

Variance of correlator output is: $\sigma_\infty^2 = \langle r_\infty^2 \rangle - \langle r_\infty \rangle^2 = \frac{\sigma^4}{N_{Nq}} (1 + r_{nrm}^2)$

The SNR is $\mathcal{R}_{sn\infty} = \frac{\langle r_\infty \rangle}{\sigma_\infty} = \frac{r_{nrm} \sqrt{N_{Nq}}}{(1 + r_{nrm}^2)} \simeq r_{nrm} \sqrt{N_{Nq}}$ for $r_{nrm} \ll 1$

Practically, N_{Nq} ranges within 10^6 to 10^{12} which is good approximation. Signal detection threshold is found from above by setting $\mathcal{R}_{sn\infty} = 1$. This shows r_{nrm} lies between 10^{-12} and 10^{-6} . The SNR of above equation can be expressed in terms of band-width as: $\mathcal{R}_{sn\infty} = r_{nrm} \sqrt{2 \Delta\nu \tau}$

Corr: Non-Nyquist Sampled, No Quantization

Let the sampling frequency is scaled as β times the Nyquist rate. Number of samples $N = \beta N_{\text{Nq}}$. Sampling interval $\tau_s = 1/(2\beta\Delta\nu)$. Samples are spaced in time by $q\tau_s$, where q is integer.

By substituting $\tau_s = 1/(2\beta\Delta\nu)$ in $R_\infty(\tau) = \frac{\sin(2\pi\Delta\nu\tau)}{2\pi\Delta\nu\tau}$ we get:

Autocorrelation $R_\infty(q\tau_s) = \frac{\sin(\pi q/\beta)}{\pi q}$

Variance $\sigma_\infty^2 = \frac{\sigma^4}{N} \left\{ 1 + 2 \sum_{q=1}^{\infty} R_\infty^2(q\tau_s) \right\}$

Signal to noise ratio (SNR)

$$\mathcal{R}_{\text{sn}\infty} = \frac{r_{\text{nrm}} \sqrt{\beta N_{\text{Nq}}}}{\sqrt{1 + 2 \sum_{q=1}^{\infty} R_\infty^2(q\tau_s)}}$$

First consider under sampling ($0 < \beta < 1$). This makes $R_\infty = 0$. Hence the denominator of last equation becomes unity. This causes the sensitivity to drop down. Now consider over-sampling, where $\beta > 1$.

This results: $\sum_{q=1}^{\infty} R_\infty^2(q\tau_s) = (\beta - 1)/2$

Here, the sensitivity is same as achieved at Nyquist rate.

Correlation: Sampled with Quantization

To refresh our memory, a few points on quantized signals are listed below:

- (i) The quantized values may or may not be close to the signal value.
- (ii) As the number of quantization levels are increased, the quantized values fall more closer to the signal.
- (iii) When the number of quantization levels becomes infinity, the quantized signal values exactly matches the signal values at those time instants.

We now discuss the correlation results of sampled but finitely quantized signals. Our focus will be on three important points:

- (i) Relationship between measured correlation with the normalized cross-correlation coefficient r_{norm} .
- (ii) Sensitivity losses.
- (iii) Extent to which the lost sensitivity may be restored by over-sampling.

Numerical subscripts denoting number of quantized levels have been used. For example, r_{n2} represents the normalized cross-correlation coefficient obtained using two level quantized signals at the correlator input.

Correlation: Two Level Quantization

Let us use a correlator with two input quantization levels represented by +1 and -1 (one-bit). The two level quantized normalized correlation coefficient r_{n2} is related to r_{nrn} (unquantized) as:

$$r_{n2} = \frac{2}{\pi} \sin^{-1} (r_{\text{nrn}}) \quad \text{This is also known as } \textit{Van Vleck relationship}.$$

Let \hat{x}_i and \hat{y}_i represent the quantized states of x_i and y_i , where i is integer which representing sample point. Correlator output r_2 by two level quantized inputs is:

$$r_2 = \frac{1}{N} \sum_{i=1}^N \hat{x}_i \hat{y}_i$$

Mean square value of r_2 is:

$$\langle r_2^2 \rangle = \frac{1}{N^2} \sum_{i=1}^N \langle [\hat{x}_i]^2 [\hat{y}_i]^2 \rangle + \frac{1}{N^2} \sum_{i=1}^N \sum_{i \neq k} \langle \hat{x}_i \hat{y}_i \hat{x}_k \hat{y}_k \rangle$$

The variance is:

$$\sigma_2^2 = \langle r_2^2 \rangle - \langle r_2 \rangle^2 \simeq \frac{1}{N} \left\{ 1 + 2 \sum_{q=1}^{\infty} R_2^2 (q\tau_s) \right\} \quad \text{if } r_{\text{nrn}} \ll 1$$

Note: q is an integer, and R_2 is autocorrelation. The points along time where the samples reside is given by $q\tau_s$. The approximation is good if $r_{n2} \ll 1$.

Correlation: Two Level Quantization

In terms of R_∞ (unquantized auto-correlation), R_2 is given as:

$$R_2(q\tau_s) = \frac{2}{\pi} \sin^{-1} \{R_\infty(q\tau_s)\}$$

The SNR \mathcal{R}_{sn2} at correlator output is

$$\mathcal{R}_{sn2} = \frac{\langle r_2 \rangle}{\sigma_2} = \frac{2r_{nrm} \sqrt{N}}{\pi \sqrt{1 + 2 \sum_{q=1}^{\infty} R_2^2(q\tau_s)}}$$

Substituting $N = \beta N_{Nq}$, we evaluate the ratio of SNR η_2 or relative sensitivity, which is the ratio of SNR of a two level quantized correlator to the SNR of an unquantized correlator. It is given as:

$$\eta_2 = \frac{\mathcal{R}_{sn2}}{\mathcal{R}_{sn\infty}} = \frac{2 \sqrt{\beta}}{\pi \sqrt{1 + 2 \sum_{q=1}^{\infty} R_2^2(q\tau_s)}}$$

Recall that N_{Nq} is the number of samples at Nyquist rate. The factor β is for studying under and over-sampling cases. The expression for R_2 from first equation gets modified as:

$$R_2(q\tau_s) = \frac{2}{\pi} \sin^{-1} \left\{ \frac{\beta \sin(\pi q/\beta)}{\pi q} \right\}$$

Sampling at Nyquist rates or less: This corresponds to $\beta = 1, 1/2, 1/3, \dots$

Here, $\sum_{q=1}^{\infty} R_2^2(q\tau_s) = 0$ We obtain: $\eta_2 \approx 0.64$.

Sampling above Nyquist rates: This results in higher values of η_2 . Dependency on the band-pass shape comes in when $\beta \geq 2$.

Correlation: Four Level Quantization

Consider a two bit system which gives $2^2 = 4$ quantization levels. Let $-n, -1, +1, +n$ represent the quantization states from lowest to highest. Inside the correlator, the product of two samples can take the values $+1, -1, +n, -n, +n^2$ and $-n^2$. In this case, the four level quantized normalized correlation coefficient r_{n4} as a function of r_{nrm} is given as:

$$r_{n4} = \frac{\langle r_4 \rangle}{\Phi_{v_0} + n^2 (1 + \Phi_{v_0})} \simeq r_{nrm} \frac{2 [(n+1)E + 1]^2}{\pi [\Phi_{v_0} + n^2 (1 - \Phi_{v_0})]} \text{ if } r_{nrm} \ll 1$$

Here, r_4 is the four level quantized correlator output, and $E = \exp(-v_0^2/2\sigma^2)$
 Φ_{v_0} is the probability that the unquantized level is restricted within $\pm v_0$ and is given as:

$$\Phi_{v_0} = \frac{1}{\sigma \sqrt{2\pi}} \int_{-v_0}^{v_0} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx = \text{erf}\left(\frac{v_0}{\sigma \sqrt{2}}\right)$$

Recall that σ^2 is the variance of the correlator output with unquantized inputs.

Let \hat{x}_i and \hat{y}_i represent the quantized states of x_i and y_i , where i is integer which representing sample point. Correlator output r_4 is:

$$r_4 = \frac{1}{N} \sum_{i=1}^N \hat{x}_i \hat{y}_i$$

Here, N is the number of samples.

Correlation: Four Level Quantization

Mean square value of r_4 is:

$$\langle r_4^2 \rangle = \frac{1}{N^2} \sum_{i=1}^N \langle [\hat{x}_i]^2 [\hat{y}_i]^2 \rangle + \frac{1}{N^2} \sum_{i=1}^N \sum_{i \neq k} \langle \hat{x}_i \hat{y}_i \hat{x}_k \hat{y}_k \rangle$$

The variance is:

$$\sigma_4^2 = \langle r_4^2 \rangle - \langle r_4 \rangle^2 = \langle r_4^2 \rangle - \rho_4^2 \{ \Phi + n^2 (1 + \Phi) \}^2$$

The variance as a function of auto-correlation R_4 is:

$$\sigma_4^2 = \frac{1}{N} [\Phi + n^2 (1 - \Phi)]^2 \left\{ 1 + 2 \sum_{q=1}^{\infty} R_4^2(q\tau_s) - r_{n4}^2 \right\}$$

Assuming $r_{nrm} \ll 1$, the SNR \mathcal{R}_{sn4} is:

$$\mathcal{R}_{sn4} = \frac{\langle r_4 \rangle}{\sigma_4} = \frac{2\rho \{ (n-1)E + 1 \}^2 \sqrt{N}}{\pi \{ \Phi + n^2 (1 - \Phi) \} \sqrt{1 + 2 \sum_{q=1}^{\infty} R_4^2(q\tau_s)}}$$

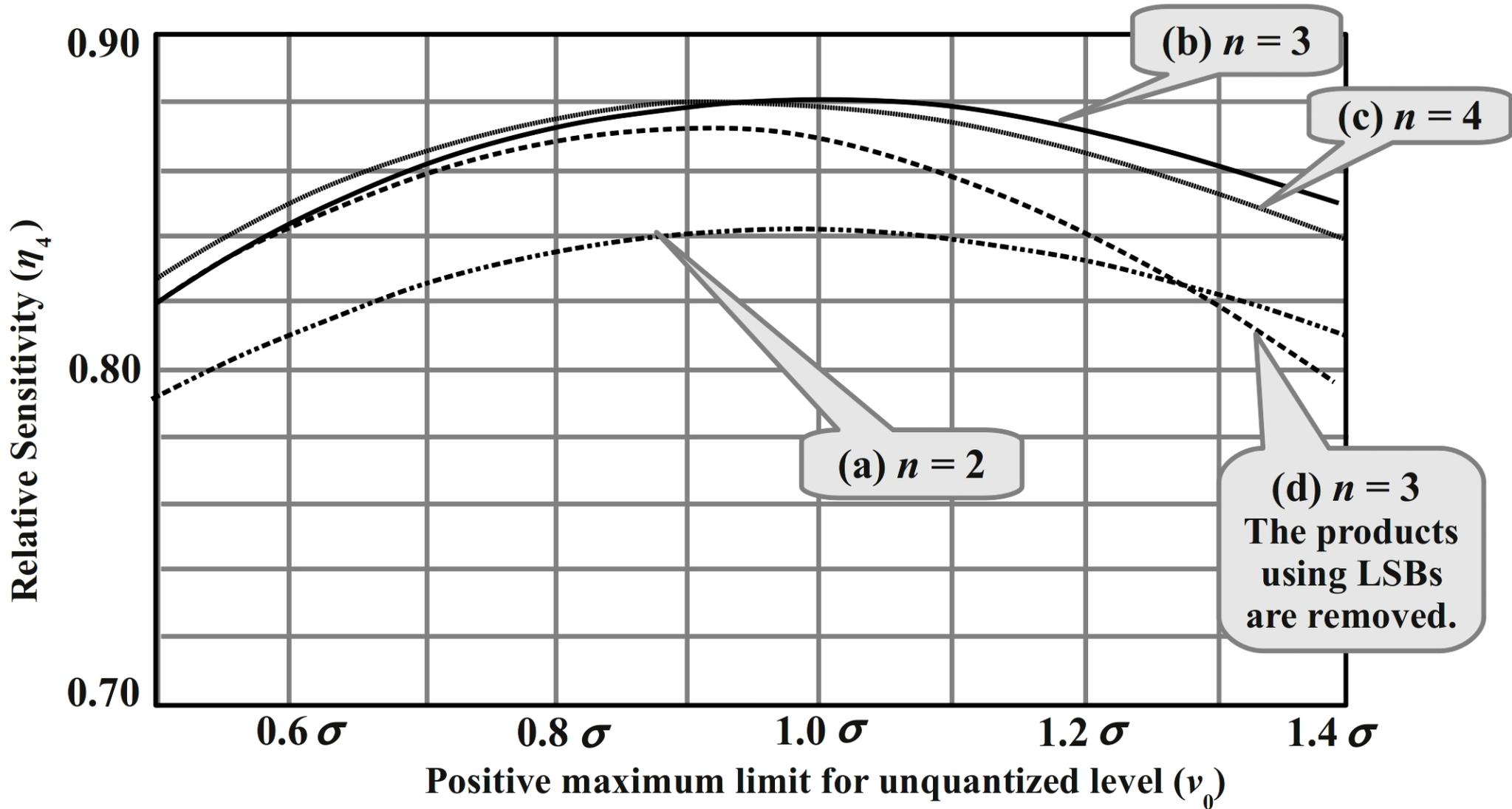
Ratio of SNRs η_4 or relative sensitivity ($\mathcal{R}_{sn4} / \mathcal{R}_{sn\infty}$) is:

$$\eta_4 = \frac{\mathcal{R}_{sn4}}{\mathcal{R}_{sn\infty}} = \frac{2 \{ (n-1)E + 1 \}^2 \sqrt{\beta}}{\pi \{ \Phi + n^2 (1 - \Phi) \} \sqrt{1 + 2 \sum_{q=1}^{\infty} R_4^2(q\tau_s)}}$$

Here, β controls the Nyquist rate.

Recall that the minimum and maximum values of quantization states in our four level quantized correlator are respectively $-n$ and $+n$. Changing the level n causes the optimized sensitivity to change. For example, when $n = 3$, the optimized sensitivity occurs with $v_0 = 0.996\sigma$ and when $n = 4$ it occurs with $v_0 = 0.942\sigma$.

Correlation: Four Level Quantization



Relative sensitivities (η_4) of a four level quantized correlator for different values of n . (a) Two bit system with $n = 2$. (b) Two bit system with $n = 3$. (c) Two bit system with $n = 4$. (d) Two bit system with $n = 3$, but low-level products deleted.

Correlation: Other Level Quantizations

Till now, we have seen the quantization levels as even numbers. Sometimes odd number of quantization levels are preferred for certain advantages. We now describe a three level quantization where one of the quantization levels is zero. The variance σ_3^2 of the correlator output r_3 is

$$\sigma_3^2 = \langle r_3^2 \rangle - \langle r_3 \rangle^2 = \frac{1}{N} (1 + \Phi)^2 \left\{ 1 + 2 \sum_{q=1}^{\infty} R_3(q\tau_s) - r_{n3}^2 \right\}$$

Here R_3 and r_{n3} respectively represent the auto-correlation and the normalized quantized correlation coefficient. Neglecting r_{n3}^2 , the relative sensitivity η_3 in terms of SNR or a three bit quantized correlator \mathcal{R}_{sn3} to the SNR of a non-quantized correlator $\mathcal{R}_{sn\infty}$ is expressed as:

$$\eta_3 = \frac{\mathcal{R}_{sn3}}{\mathcal{R}_{sn\infty}} = \frac{\langle r_3 \rangle}{\sigma_3 \mathcal{R}_{sn\infty}} = \frac{2 \sqrt{\beta} E^2}{\pi (1 + \Phi) \sqrt{1 + 2 \sum_{q=1}^{\infty} R_3^2(q\tau_s)}}$$

Here Φ is obtained as: $\Phi = \frac{1}{\sigma \sqrt{2\pi}} \int_{-v_0}^{v_0} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx = \text{erf}\left(\frac{v_0}{\sigma \sqrt{2}}\right)$

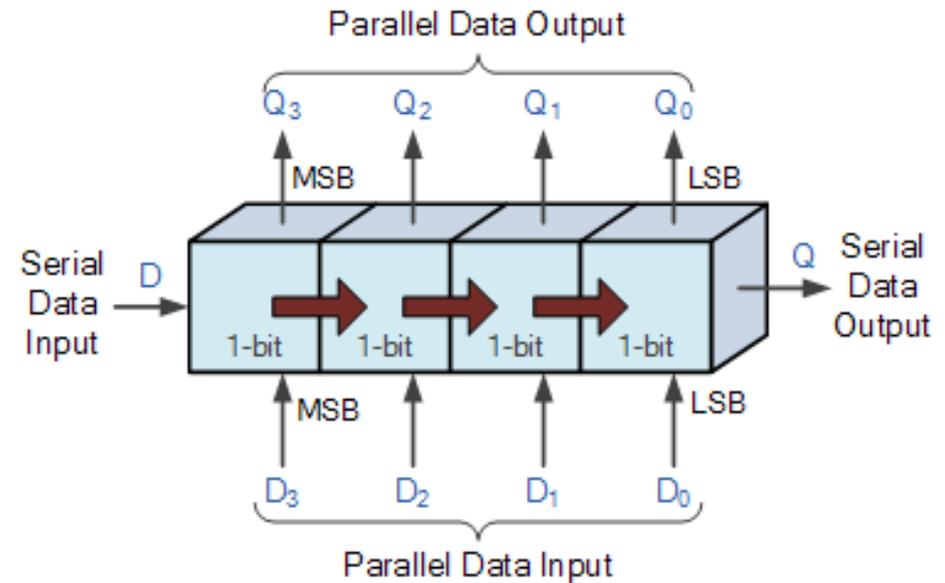
Note: Sensitivity does not improve very much for quantization levels greater than four.

Correlator: Principle of Digital Time Delay

We know that geometrical delay τ_g in a correlator system is compensated by generating an instrumental delay τ_i . In digital systems the delay is applied in discrete time steps, usually in integer multiples of sampling period, i.e. $k \tau_s$, where k is an integer and τ_s is the sampling period. The integer k controls the delay period.

Shift register: Each clock pulse shifts any cell's data bit to the adjacent right cell. Data can be entered and taken out in following modes:

- (i) Serial-in to Parallel-out.
- (ii) Serial-in to Serial-out.
- (iii) Parallel-in to Serial-out.
- (iv) Parallel-in to Parallel-out.



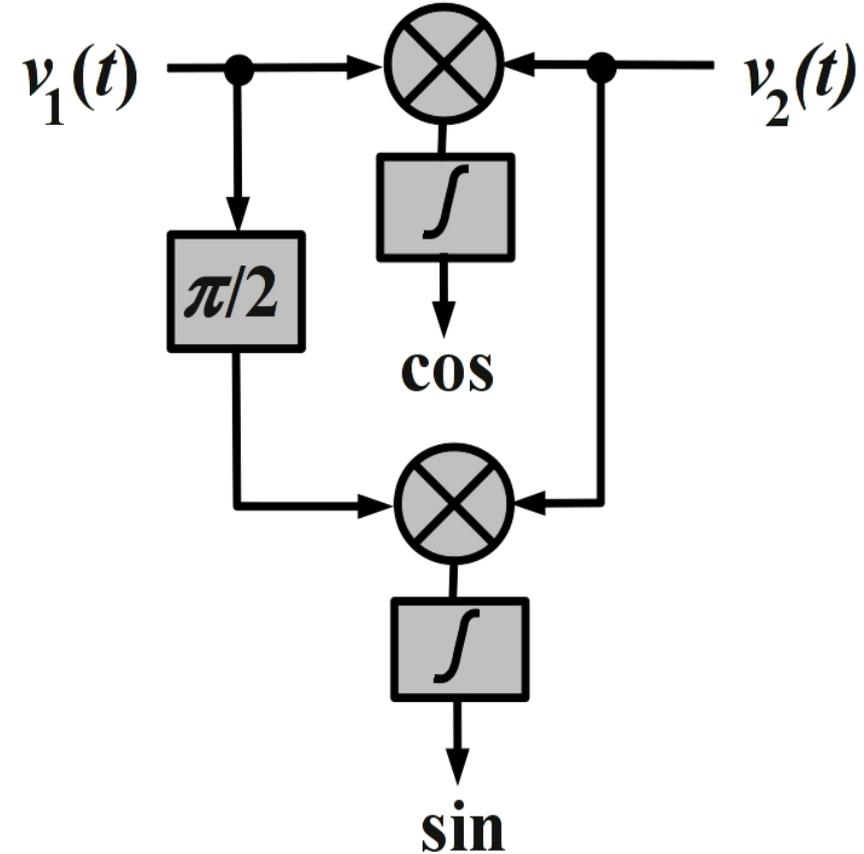
Shift registers may be used for generating delay. In a shift register, the first cell is assigned 1 and the remaining cells are assigned 0. A time period of τ_{clk} is required for moving a cell's data to its adjacent cell. For convenience, τ_{clk} is kept same as sampling period τ_s . Currently, digital ICs are available which are software controlled and generates customized delays.

Principle of Digital Quadrature Phase Shift

The **sin** component in a complex correlator is obtained by applying a 90° phase shift in one of the signals before multiplying them.

Analog correlator - Hilbert transform of **cos** signals gives **sin** signals.

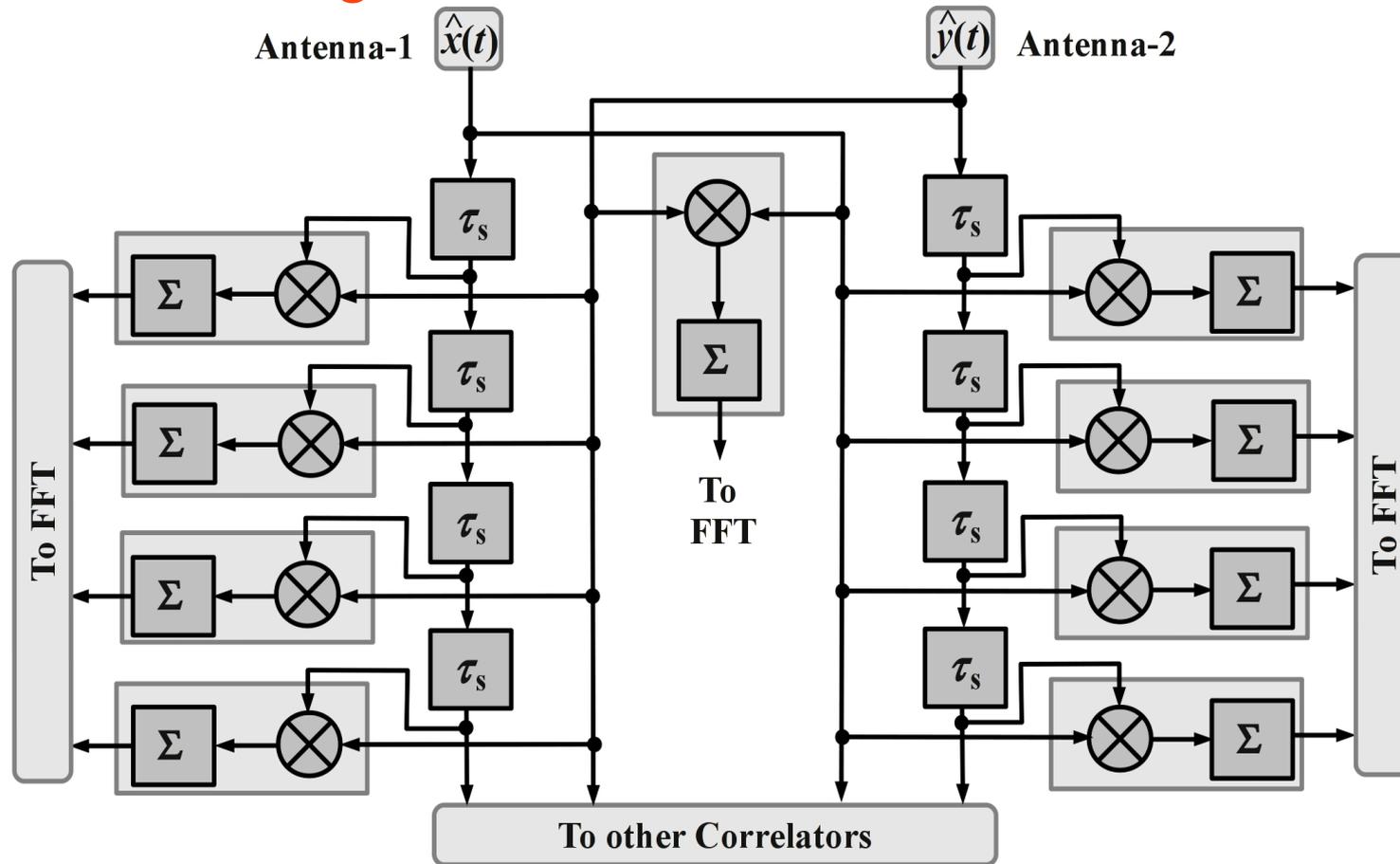
Digital correlator - Hilbert transform using digital circuits gives a limited accuracy because of limited number of samples. SNR reduces by a few percent.



Reasons: Summation process in digital convolution increases the number of bits in the data. To reduce complexity, lower order bits are discarded resulting further quantization loss. Effectively, **sin** output suffers from spectral distortion and reduced SNR as compared to **cos**. While designing broad-band systems, these are more serious considerations where the high data rate leaves a scope only for simple processing.

XF Correlator

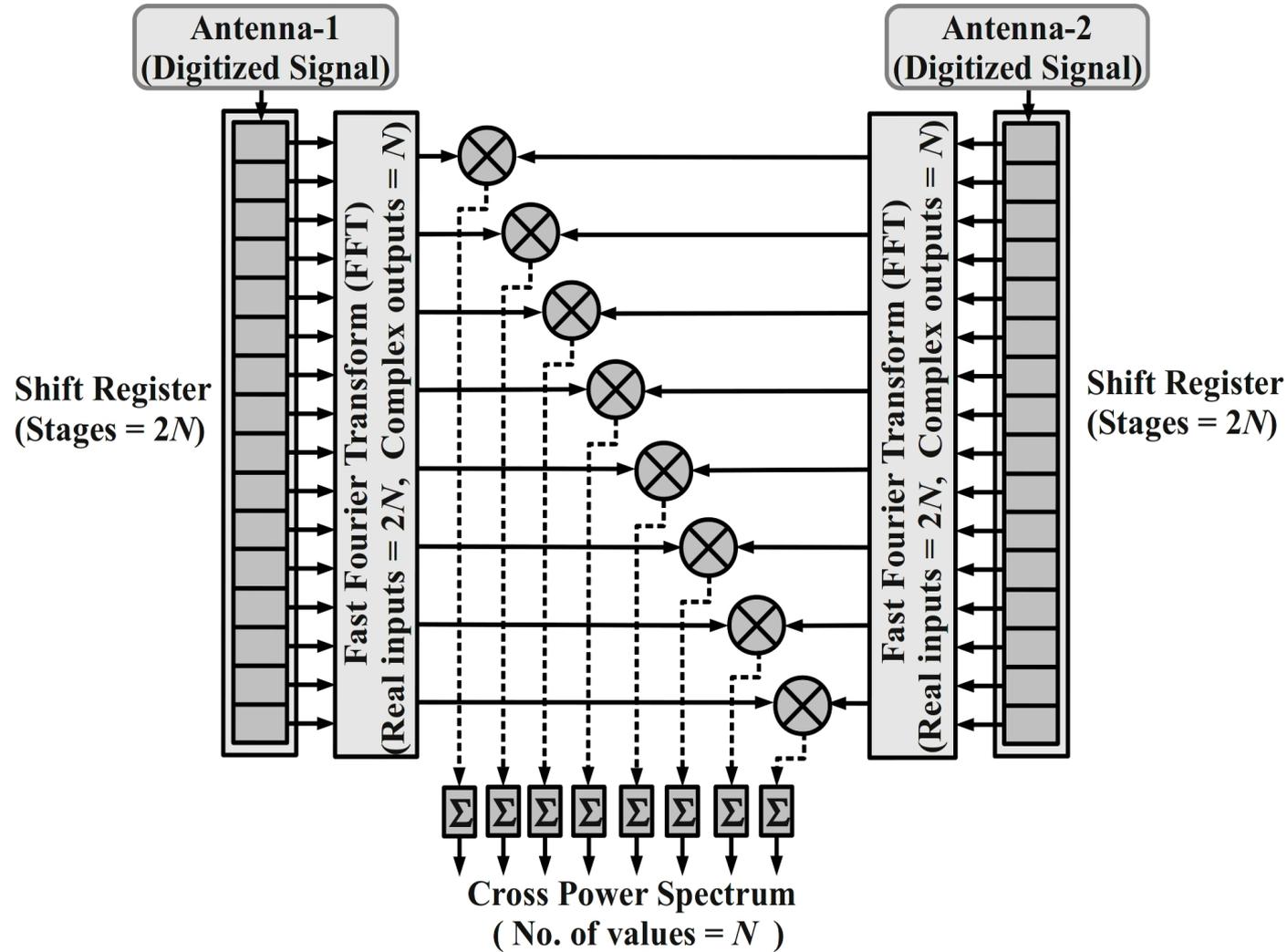
XF is a correlator symbol which indicates that the correlation (X) is done in time domain followed by Fourier transformation (F) to frequency domain. This is also known as lag correlator.



A XF (lag correlator) using two antennas. Delay is applied in integral multiples of sampling period τ_s . FFT is applied on correlator output. The left, middle and right set of outputs are respectively correlation measures between (i) \hat{y} and delayed \hat{x} , (ii) \hat{y} and \hat{x} , and (iii) \hat{x} and delayed \hat{y} .

FX Correlator

FX is a correlator symbol which indicates Fourier transform (F) is performed first and then correlation (X) is followed. Input bit streams from antennas are converted to frequency spectra by a real time FFT. Each of the frequency channels are then multiplied and integrated in complex form.



A FX correlator using two antennas. Digitized signals are fed into the shift registers, whose output undergoes FFT at intervals of $2N$ sample periods. Correlations are made between one signal with the complex conjugate of the other. For an array of n_a antennas, each of the FFT outputs are split $(n_a - 1)$ ways for combining with complex amplitudes from the other antennas.

Very Long Baseline Interferometry (VLBI)...

VLBI came from the requirements of higher resolution astronomical images at radio. Interferometer's angular resolution at wavelength λ is $\lambda/(2d\cos\theta)$, where d is baseline distance. Highest resolution possible on Earth is by setting d equal to Earth's diameter (antennas on opposite sides of Earth).

Data from radio arrays all over the world are calibrated in time and phase and combined. Due to phase-instability in fiber links beyond 200 km, a single array data set is difficult to obtain. Thus extreme phase-stable LO signals (using atomic clocks) are used. Individual data sets marked with precise timing are recorded independently. These are collectively matched and processed (mutual correlations are generated from data). The process involves correction of differential Doppler shifts due to Earth rotation and clock rate offsets.

Recently it has become possible to connect the VLBI radio telescopes in real-time employing the local time references using a technique known as **e-VLBI**. In Europe, six radio telescopes of the European VLBI Network (EVN) are connected with Gigabit per second links via their National Research Networks and the Pan-European Research Network GEANT2, and the first experiments using this technique was successfully conducted.

Assignment Problems-I

1. Two parabolic dish antennas of 2 m diameter are separated by 20 m. They form an interferometer. If the operating frequency is 150 MHz, calculate the angular resolution.

Hint: $\alpha'' = \frac{\lambda}{2d}$

2. An interferometer forms a fringe pattern on the screen due to light from a distant star. The maximum intensity measured is found to be twice the minimum intensity. Calculate the magnitude of visibility.

Hint: $|\mathcal{V}| = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$

3. An optical interferometer uses a wave band-width of 1 nm. Find the coherence length at a wavelength of 500 nm. Compare this with a radio interferometer working at 610 MHz having a band-width of 32 MHz.

Hint: $l_c = \frac{c}{n_r \Delta\nu} \approx \left(\frac{1.4}{\pi n_r} \right) \frac{\lambda^2}{\Delta\lambda}$

4. Explain the additive radio interferometer and its fringe pattern using suitable diagrams.

Assignment Problems-II

5. Using diagrams explain a two element multiplicative radio interferometer and its fringe pattern.
6. What is meant by fringe stopping center of a radio interferometer?
7. What can be the maximum integration time of the correlator? Can it be increased infinitely for a moving source?
8. Give diagrammatic representations of (i) a simple correlator and (ii) complex correlator. Explain these using equations.
9. Give the number of cross-correlator outputs obtained from an array of 30 antennas.
10. For a correlator array of 30 antennas, what would be the percentage loss in power if the self terms are eliminated?

Hint:

$$\kappa_c = 1 - \frac{1}{n_a}$$

Assignment Problems-III

11. Super heterodyne receiving systems are used in a multiplicative interferometer and delay correction is done in IF. Explain the term natural fringe frequency giving an expression.

Hint:
$$\nu_{fringe} = \nu_{LO} \frac{d\tau_g}{dt}$$

12. If the number of IF stages is n and if only the LSB are chosen, for what values of n does the spectrum flip?

13. If r_{usb} and r_{lsb} are respectively the upper and lower side-band correlator responses of a single IF conversion receiver, what is the correlator response if both side-bands are used? Is there any gain in sensitivity?

14. For above problem, draw tentative correlator responses as a functions of non-zero $\Delta\tau$, where $\Delta\tau = \tau_g - \tau_i$ (difference between geometrical and instrumental).

15. Tentatively draw the complete noise spectra that will be available within a period equal to a single integration period of the correlator if integration is not performed and explain it in details.

Assignment Problems-IV

16. An analog interferometer consisting of two antennas A and B are used to observe a radio source. If the system temperatures using A and B are respectively 35 K and 40 K respectively and their respective antenna temperatures are 45 K and 43 K, find the SNR. Assume the integration bandwidth as $1/(10\text{s}^{-1})$ and IF system band-width as 32 MHz.

Hint:

$$\mathcal{R}_{\text{sn}} = \sqrt{\frac{T_{Am}T_{An}}{T_{Sm}T_{Sn}}} \sqrt{\frac{\Delta\nu_{\text{IF}}}{\Delta\nu_{\text{LF}}}}$$

17. An interferometer shows a system temperature of 40 K and antenna temperature of 50 K. If the IF band-width is 32 MHz and integration time of 16 sec, find the SNR for a (i) analog system, and for a (ii) digital system with quantization efficiency 70%.

Hint:

$$\mathcal{R}_{\text{sn}} = \frac{T_A \eta_Q}{T_S} \sqrt{\frac{\Delta\nu_{\text{IF}} \tau_a}{2}}$$

18. Calculate the minimum detectable flux density produced by an unresolved source located at the phase reference center of an interferometer, given: diameter of the dishes as 45 m, aperture efficiency as 40%, system temperature as 50 K, integration time as 16 seconds, IF band-width as 32 MHz and quantization efficiency as 70%.

Assignment Problems-V

Hint: Calculate the effective aperture area, then use the following equation.

$$S_{\text{crr}} = \frac{\sqrt{2} k T_S}{A_e \eta_Q} \frac{1}{\sqrt{\Delta\nu_{\text{IF}} \tau_a}}$$

19. Explain the sampling theorem for discretizing analog signals without loss of information.
20. Explain the band-pass sampling technique.
21. If x_s and y_s are the instantaneous signals from the two antennas of an interferometer, derive expression for the normalized cross-correlation coefficient using variance when mean is zero. Justify your derivation.
Hint: Begin with the following equation:

$$r_{\text{nrm}} = \frac{\langle x_s(t) y_s(t) \rangle}{\sqrt{\langle [x_s(t)]^2 \rangle \langle [y_s(t)]^2 \rangle}}$$

22. Explain the basic principle behind generating digital time delay. What is the minimum delay period that can be produced? Give a scheme for delay generation.
Hint: Shift registers.

Assignment Problems-VI

23. Explain the basic technique used for generation of digital quadrature phase shifted output from a digital correlator.
24. Using a diagram explain the XF (lag) correlator.
25. Using a diagram explain the FX correlator.
26. Write short notes on VLBI.

THANK YOU